Optimal Fiscal Policy with Endogenous Time Preference

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Abstract

This paper studies the role of Ramsey taxation under the assumption that the individual rate of time preference is determined by the publicly-provided social level of education. We show how intertemporal complementarities of aggregate human capital can generate multiple equilibria and we examine the role of endogenous fiscal policies in equilibrium selection. Our analysis implies a lower optimal government size due to the effect of human capital on time preference.

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1 Introduction

In his seminal work, Ramsey (1927) took into account agents' equilibrium reactions in forming the government's optimal fiscal policy. This second-best approach has been extensively revitalized in capital accumulation models of unique equilibrium and exogenous time preference (Lucas and Stockey, 1983; Chamley, 1986; Judd, 1987; Lucas, 1990; Jones et al., 1993). The present paper introduces the role of second-best Ramsey environment in equilibrium selection under the presence of multiple equilibria generated by intertemporal complementarities of human capital in the formation of time preference. Our theoretical framework provides a new role for the conduct of optimal fiscal policy under indeterminacies and poverty traps (Ben-Gad, 2003; Park and Philippopoulos, 2004; Park, 2009; Agénor, 2010). We also analyze some interesting policy implications as the standard productive effects of optimal taxation are altered (Barro, 1990; Futagami et al., 1993; Glomm and Ravikumar, 1997; Turnovsky, 2000).

The starting point of our analysis is a general equilibrium model in which the rate of time preference depends positively on the economy-wide consumption level and negatively on the aggregate human capital stock. Since Uzawa (1968), several studies have investigated the effects of individual consumption on the time preference rate; see Obstfeld (1981, 1990), Mendoza (1991), Shin and Epstein (1993), Palivos et al. (1997), Drugeon (1996, 2000), Uribe (1997), Schmitt-Grohé (1998), Stern (2006) and Chen et al. (2008). In turn, Epstein and Hynes (1983), Schmitt-Grohé and Uribe (2003), and Choi et al. (2008) have, among others, endogenized the rate of time preference to aggregate consumption in variants of general equilibrium models. These models have highlighted the importance of endogenous time preference for the dynamic patterns of consumption. Yet, very little is known about the policy aspects of endogenous time preference to fiscal policy, with the exception of Agénor (2010) who assumes that public health services lower impatience.

We introduce optimal fiscal policy in the form of Ramsey taxation in this strand of the literature by assuming that the publicly-provided aggregate human capital stock also affects time preference. The main underlying idea in the paper is that agents are less impatient in a more educated surrounding environment. This point goes back to Strotz (1956), who had noticed that discount functions are formed by teaching and social environment, and was re-raised by Becker and Mulligan (1997), who argued that schooling and other social activities in "future-oriented capital" focus agents' attention to the future.[1] Doepke and Zilibotti (2008) further explored the role of parental time invested in patience to develop a theory of preference formation and explain the historical reversals in economic fortunes.

Existing empirical evidence suggests that education strongly affects patience by rendering agents less impulsive to choices that tend to overweight rewards in close temporal proximity. Fuchs (1982) was the first study that attempted to investigate empirically the association between time preference and education, and showed that there is a positive link between patience and years of schooling. Lawrance (1991) has found that nonwhite families without a college education have time preference rates that are about seven percentage points higher than those of white. Similarly, Harrison et al. (2002) have shown on a sample of Danish households that highly educated adults have subjective discount rates that are roughly two thirds compared to those who are less educated. Khwaja et al. (2007) have found that the years of education affect negatively the degree of impulsivity defined as the measure of an individual's ability to set goals and to exercise self-control. Recently, Meier and Sprenger (2010) and Perez-Arce (2011) report that college education is significantly associated with time preference and Bauer and Chytilova (2009, 2010) estimate that an additional year of schooling in Ugandan villages lowered significantly the discount rate.[2] An inspection of available cross-country data confirms that indeed there is a negative association between time preference and education, portrayed in Figure 1 that depicts estimates of time preference for 32 countries, obtained from Wang et al. (2010), and the corresponding years of schooling (Barro et al., 2001).[3]

In an endogenous growth framework, we first examine the equilibrium properties of the decentralized economy and we show that there can be one or two positive equilibrium growth rates, whereas global indeterminacy can be accompanied by local indeterminacy. The central mechanism that drives these results arises from two counterbalancing channels. First, a rise in human capital financed by an increase in the tax rate lowers the rate of time preference, causing savings to increase

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[1] Perhaps the most prominent illustration of the fiscal policy aspects associated with this issue concerns the causes of the high savings rate observed in post-war Japan. Horioka (1990) and Sheldon (1997, 1998) have attributed this behavior, among other factors, to an array of public policies implemented through educational programmes that promoted the virtues of patience and thrift.

[2] In addition, an indirect channel regarding the impact of human capital on impatience may come through income and wealth: Hausman (1979) and Samwick (1998) have found that discount rates are inversely related to income level, and Horowitz (1991) and Pender (1996) have reported that discount rates decline with wealth.

[3] Notice that the empirical association illustrated in Figure 1 does not necessarily imply a causal relationship between time preference and human capital. The dynamics of our general equilibrium setup are able to capture the endogeneity between these two variables.

and as a result the economy can attain higher growth. This in turn increases the tax base, raises public expenditures on education and hence fuels further growth. On the other hand, the rise in taxation decreases private savings, which increases the rate of time preference in the economy due to the rise in aggregate consumption and hence lowers growth. A lower growth rate in turn lowers the tax base that finances human capital formation leading to even higher time discounting. We establish necessary and sufficient conditions for the existence of a unique or multiple (two) balanced growth paths (BGPs) and we provide numerical examples that highlight the intuition behind these mechanisms.

Equilibrium selection is then addressed by endogenizing fiscal policy in the context of second-best Ramsey allocation. We demonstrate, first, how the government's objective can determine the available set of policy instruments (Atkinson and Stiglitz, 1980) and, second, its importance in the implementation of additional restrictions on private decisions that can lead the decentralized economy to a unique BGP. Typically global and local indeterminacy at the decentralized equilibrium implies that the rational expectations equilibria involve random variables, which are unrelated to the economy's fundamentals and are driven by individual beliefs. However, in a second-best (Stackelberg) environment the government can obtain information through the agents' reaction function and consequently impose additional restrictions on the tax rate and the endowment allocation through commitment, in order to drive the economy to a unique equilibrium. This policy is feasible under a state-dependent taxation rule that is linked to aggregate endowments and internalizes intertemporal complementarities of human capital, which fuel multiplicity of equilibria.

Some fiscal aspects of our work should be stressed in comparison with the existing literature. First, a connection can be established to earlier studies that have investigated, using models in which externalities generate multiple growth paths, the role of public policy in eliminating the poverty trap and selecting the desired equilibrium (Matsuyama, 1991; Boldrin, 1992; Rodrik, 1996). However, these studies have examined how government intervention can affect the set of equilibria that exist under laisser-faire, without explicitly specifying the government's objective and by using an exogenous tax rate. Endogenizing government policy becomes an interesting task because the tax rate is now an endogenous variable, which depends on the actions by private agents, yet can create rather than eliminate coordination failures and strategic uncertainty, thus causing multiple equilibria (Cooper, 1999). In particular, in Park and Philippopoulos (2004) and Park (2009) multiple equilibria in the decentralized economy are the outcome of endogenous policy indeterminacy in the form of multiple tax rates. In an OLG endogenous growth model, Glomm and Ravikumar (1996) show that there may be multiple equilibrium paths when public policy is endogenous. Furthermore, Ben-Gad (2003) has shown that a sufficient degree of capital taxation in a Lucas–Uzawa endogenous growth model or a combination of factor taxation and external effects can trigger equilibrium indeterminacy in the form of many (more than two) balanced growth paths. In comparison to these studies, multiplicity can arise here for any feasible range of exogenously set tax rate, whereas endogenous Ramsey taxation is not only determinate (unique tax rate) but can also lead the market economy to the desired growth regime under a state-depended taxation rule.

Second, other studies have also examined the possibility of equilibrium indeterminacy with endogenous time preference. Drugeon (1996) has studied the possibility of multiple steady states when the production technology exhibits increasing returns and the investment technology is non-linear. In contrast, the instantaneous utility and production functions adopted here satisfy the standard concavity assumptions and the investment technology is linear. Chen (2007) assumes that time preference depends on individual past consumption through habit formation, which forms an internal, rather than external, intertemporal complementarity resulting in multiple equilibria that arise by the interactions of consumption levels at different time periods. Recently, Agénor (2010) explores the network effects of an exogenous rise in public infrastructure, which can facilitate, through the rise in health services and patience, the shift from a low-savings poverty trap to a steady state characterized by high growth. In the present paper we point out instead the endogenous fiscal policy impacts on the optimal dynamic individual choice, which now depends on current and lagged human capital formation decisions that can generate multiple equilibria and propagate growth effects over time.

Finally, a public policy implication of our analysis is that the endogeneity of time preference alters the standard effects of taxation. For instance, if the intertemporal elasticity of substitution is sufficiently low, the tax rate has to be lower than the standard growth-maximizing taxation rule, which states that the tax rate on output should equal the elasticity of public capital in the production function (Barro, 1990; Futagami et al., 1993; Glomm and Ravikumar, 1997). This effect is triggered by the lower marginal cost of taxation due to the favorable effect of endogenously-chosen taxation on human capital financing and, in turn, on the rate of time preference and economic growth as the tax base increases.

Our analytical results have some novel policy implications for economic performance as it is argued that active public policies in sectors like education are crucial in boosting growth, particularly in countries that face development traps. Given that countries with similar structural characteristics often seem to display divergent economic behavior, our findings suggest an additional generating mechanism of "low-growth" equilibria. This stems from the linkage between endogenous time discounting and productive fiscal policy, with the latter now operating through the demand, rather than the supply, side of the economy by forming the patience of consumers. In turn, our results on the role of second-best fiscal policy in driving the economy to a "high-growth" path, albeit highly stylized, indicate the importance of active policymaking in determining the long-run performance of the economy by affecting individual patience.

The rest of the paper is structured as follows. Section 2 sets up and solves the optimization problem of households and firms, and studies the steady state and the dynamic properties of the decentralized economy. Sections 3 and 4 analyze the role of Ramsey second-best and growth-maximizing taxation in equilibrium selection. Finally, section 5 concludes the paper.

2 The Competitive Decentralized Equilibrium

2.1 The basic model

Consider an economy with a constant number of infinitely-lived agents that consume a single good. We assume that the rate of time preference, ρ, is not a positive constant, as in standard growth theory, but is endogenously determined by aggregate consumption, C, and aggregate human capital, H. Each household seeks to maximize intertemporal discounted utility given by:

$$\int_0^\infty u(c) \exp\left[-\int_0^t \rho(C_s, H_s) ds\right] dt \tag{1}$$

with instantaneous utility function of the form $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, where $0 < \sigma \leq 1$, subject to the initial asset endowment $A(0) > 0$ and the income resource constraint:

$$\dot{A} = rA + w - c \tag{2}$$

where A denotes per capita financial assets, c denotes per capita consumption, r and w denote the market interest rate and the wage rate respectively.[4]

The time preference function has the following properties:

Assumption 1 $\rho(C,H) \geq \check{\rho} > 0$.

Assumption 2 $\rho'_C \geq 0$ *and* $\rho'_H \leq 0$.

Assumption 3 $\rho(C,H) = \rho(\frac{C}{H})$.

Assumption 1 shows that the rate of time preference is positive implying that there exists a lower bound denoted by $\check{\rho}$. By Assumption 2 the rate of time preference depends positively on aggregate consumption and negatively on aggregate human capital.[5] As in the rest of the literature with endogenous time preference, we assume that as agents consume more the value of current consumption increases (Epstein and Hynes, 1983; Schmitt-Grohé and Uribe, 2003; Choi et al., 2008). In addition, we assume that the higher the human capital stock in the economy the more patient is the agent and willing to forego current consumption (Becker and Mulligan, 1997). Assumption 3 implies homogeneity of the rate of time preference to the ratio of consumption to human capital, which is required for the rate of time preference to be bounded at the steady-state (Palivos et al., 1997) and for the utility function to be consistent with balanced growth (Boyd, 1990; Dolmas, 1996). Note that Assumptions 2 and 3 imply that the derivative of time preference to the ratio of consumption to human capital, $\frac{\partial\rho}{\partial(\frac{C}{H})} \equiv \rho'(\cdot)$, is positive.

In the supply side of the economy there exists a continuum of perfectly competitive homogenous firms, normalized to unity, that seek to maximize profits. Each firm i uses physical capital, K_i, and labor, L_i, under the following production technology:

$$Y_i = K_i^a (hL_i)^{1-a} \tag{3}$$

where $0 < a < 1$ denotes the share of physical capital in the production function, Y_i denotes individual output, and h denotes labor productivity. The law of motion for the physical capital

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[4] Throughout the paper, the time subscript t is omitted for simplicity of notation. Note that positive felicity is guaranteed only for $0 < \sigma < 1$. We also include here the logarithmic utility case ($\sigma = 1$) to allow for comparisons in our simulations with this extensively used specification.

[5] We retain the equality sign in our assumptions to allow, first, for comparisons with the case of constant rate of time preference and, second, for the impatience function to be consistent with a BGP along which the time preference is constant.

stock is given by:

$$\dot{K}_i = I_i - \delta_K K_i \tag{4}$$

where I_i denotes investment in physical capital and δ_K denotes the physical capital depreciation rate.

Following Glomm and Ravikumar (1997), we assume that human capital is provided by the public sector and serves as an input in the production function. In particular, labor productivity depends on the average human capital stock and is given by:

$$h = \frac{H}{L} \tag{5}$$

where L denotes the aggregate labor force. The law of motion for the human capital stock is then given by:

$$\dot{H} = vI_H - \delta_H H \tag{6}$$

where I_H denotes public expenditures on education, δ_H denotes the human capital depreciation rate and v is a scale parameter capturing the technology of education. Following Turnovsky (1996, 2000), we assume that the government sets its expenditures as a fixed fraction of output and imposes a flat tax rate on output, τ, to finance spending on human capital according to a balanced budget policy given by:[6]

$$I_H = \tau Y \tag{7}$$

2.2 The reduced model and balanced growth

We can now define the Competitive Decentralized Equilibrium (CDE) of the economy in order to analyze its properties.

Definition 1 *The CDE of the economy is defined for the exogenous policy instruments τ, factor prices r, w, and aggregate allocations K, H, I_H, L, C, such that*

i) Individuals solve their intertemporal utility maximization problem by choosing c and A, given τ and factor prices.

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[6] It is straightforward to show that we could obtain exactly the same results if the flat tax rate was imposed on labor and capital income because of the Cobb-Douglas production technology.

ii) Firms choose L_i *and* K_i *in order to maximize their profits, given factor prices and aggregate allocations.*

iii) All markets clear and in the capital market $A = \frac{K}{L}$ *(per capita assets held by agents equal capital stock per capita)*

iv) The government budget constraint holds.

The CDE is then defined by (i)-(iii) under the aggregation conditions $\int\limits_0^1 K_i = K$, $\int\limits_0^1 L_i = L$.

The per capita growth rate of consumption in the CDE is given by:

$$\frac{\dot{c}}{c} = \frac{1}{\sigma}[r - \rho(\cdot)] \tag{8}$$

The first-order conditions of the firms' profit maximization problem are given by $r = (1 - \tau)a\left(\frac{K_i}{hL_i}\right)^{a-1} - \delta_k$ and $w = (1-\tau)(1-a)\left(\frac{K_i}{L_i}\right)^a h^{1-a}$, and state that the marginal productivity of capital and labor have to equal respective factor prices. Using the equilibrium conditions for homogenous and symmetric firms $L_i = L$ and $K_i = K$, and assuming for the rest of the paper without loss of generality that $\delta_K = \delta_H = \delta$, the equilibrium growth rates of aggregate consumption, aggregate physical and human capital stocks are given by the following equations:

$$\frac{\dot{C}}{C} = \frac{1}{\sigma}\left[a(1-\tau)\left(\frac{K}{H}\right)^{a-1} - \rho(\cdot) - \delta\right] \tag{9}$$

$$\frac{\dot{K}}{K} = (1-\tau)\left(\frac{K}{H}\right)^{a-1} - \frac{C}{K} - \delta \tag{10}$$

$$\frac{\dot{H}}{H} = v\tau\left(\frac{K}{H}\right)^{a} - \delta \tag{11}$$

The transversality condition for this problem is given by:

$$\lim_{t\to\infty}\frac{K(t)}{C(t)}\exp\{-\int\limits_0^t \rho\left(\frac{C(s)}{H(s)}\right)ds\} = 0 \tag{12}$$

Equations (9), (10), (11) summarize the dynamics of our economy. At the BGP consumption, physical and human capital grow at the same rate, $\frac{\dot{C}}{C} = \frac{\dot{K}}{K} = \frac{\dot{H}}{H} = g^{CDE}$. This result can be easily obtained by investigating the equilibrium growth rates of these variables separately. In particular, for the steady-state consumption growth rate, given by (9), to be constant, both K and H have to grow at the same constant rate, say $g_K^{CDE} = g_H^{CDE} = g^{CDE}$. Since the steady-state ratio of human to physical capital will be constant, the equilibrium growth rate of consumption, g_C^{CDE}, will be constant too. Then, by inspection of (10), in order for the growth rate of physical capital to be constant we need that $g_K^{CDE} = g_H^{CDE}$ and $g_C^{CDE} = g_K^{CDE}$ (for $\frac{K}{H}$ and $\frac{C}{K}$ to be constant respectively). These conditions imply that $g_C^{CDE} = g_K^{CDE} = g_H^{CDE} = g^{CDE}$, which also satisfies (11) as well as the transversality condition (12). Hence, the necessary condition for the existence of a BGP in this economy is that all variables grow at the same rate, g^{CDE}.

We can now derive the equilibrium growth rate of the economy. We first define the following auxiliary stationary variables, $\omega \equiv \frac{C}{K}$ and $z \equiv \frac{K}{H}$. It is straightforward to show that the dynamics of (9)-(11) are equivalent to the dynamics of the following system of equations:

$$\frac{\dot{\omega}}{\omega} = (\frac{a}{\sigma} - 1)(1-\tau)z^{a-1} + \omega - \frac{1}{\sigma}\rho(z\omega) - (\frac{1}{\sigma} - 1)\delta \tag{13}$$

$$\frac{\dot{z}}{z} = (1-\tau)z^{a-1} - \omega - v\tau z^{a} \tag{14}$$

The following Proposition determines the properties (existence and uniqueness) of the BGP at which $\frac{\dot{\omega}}{\omega} = \frac{\dot{z}}{z} = 0$.

Proposition 1 *The growth rate of the economy at the BGP with endogenous time preference to the ratio of aggregate consumption to human capital is determined by (11) and, for given parameter values and tax rate, is given by:*

$$g^{CDE} = v\tau\bar{z}^{a} - \delta$$

provided that there exists $\bar{z} > 0 : \Phi(\bar{z}) = (\frac{a}{\sigma})(1-\tau)\bar{z}^{a-1} - v\tau\bar{z}^{a} - \frac{1}{\sigma}\rho(\bar{z} \cdot \bar{\omega}(\bar{z})) - (\frac{1}{\sigma} - 1)\delta = 0$ *and* $\bar{\omega}(\bar{z}) = (1-\tau)\bar{z}^{a-1} - v\tau\bar{z}^{a} > 0$, *where* $\bar{\omega}$ *and* $\bar{z}$ *are the steady-state values of* ω *and* z *respectively. We distinguish the following cases:*

Case 1: A sufficient condition for the existence of a unique well-defined physical to human capital ratio, which corresponds to a positive equilibrium growth rate, is $(\frac{a}{\sigma} - 1)(1-\tau)^{a}\tau^{1-a}v^{1-a} - (\frac{1}{\sigma} - 1)\delta < \frac{1}{\sigma}\breve{\rho}$.

Case 2: A necessary condition for the existence of two well-defined physical to human capital ratios, which correspond to two positive equilibrium growth rates, is $(\frac{a}{\sigma} - 1)(1-\tau)^{a}\tau^{1-a}v^{1-a} - (\frac{1}{\sigma} -$

$1)\delta \geq \frac{1}{\sigma}\check{\rho}$. *This condition is also sufficient if* $\rho^{''}(.) \leq 0$ *and* $-\frac{1}{\sigma}\rho'(0) > (\frac{a\tau v^2}{1-\tau})((\frac{a-1}{\sigma}) - 1)$.

Proof. See Appendix 1. ∎

Proposition 1 states that when the rate of time preference in the economy depends on the ratio of aggregate consumption to human capital there can be a unique or multiple (two) equilibrium growth rates. Hence, although the instantaneous utility and production technology functions satisfy the standard concavity assumptions, the existence of a unique positive steady-state growth rate is not guaranteed under the assumption that the aggregate human capital affects the impatience rate of agents.

The central mechanism that drives multiplicity arises from two counterbalancing channels. Consider the construction of an equilibrium path through a rise in the tax rate in order to finance human capital accumulation. This decreases private savings and increases the rate of time preference in the economy due to the rise in aggregate consumption, thus lowering growth. A lower growth rate in turn lowers the tax base that finances public investment in human capital leading to even higher time discounting. On the other hand, by increasing the tax rate the government increases the level of human capital expenditures in the economy. Thus, the rate of time preference falls, savings propensity increases and the economy can attain higher growth, which in turn increases the tax base and raises public expenditures on education and growth. In other words, apart from the standard relation between human capital and growth through the production function, there is an intertemporal complementarity between human capital, time preference and growth through the Euler equation (8), setting off a virtuous growth cycle.

The final outcome will depend on the structural parameters of the economy. Case 2 of Proposition 1 (multiplicity of equilibria) is more likely to arise when the inverse of the intertemporal elasticity of substitution, σ, is sufficiently low and the elasticity of human capital in the production function is sufficiently high. For instance, assuming a zero depreciation rate of human capital, it is straightforward to show that when $\sigma < \alpha$ the second channel dominates for any tax rate. This happens because the standard dynamic mechanism of intertemporal substitutability between the savings rate and the rate of return on capital that preserves a unique BGP fails, thus giving rise to equilibrium indeterminacy.

The following numerical example highlights our analytical result for a linear time preference function, $\rho(\frac{C}{H}) = b * (\frac{C}{H}) + \check{\rho}$, which is used for computational tractability.

Example 1 *Consider a linear time preference function that satisfies Assumptions 1-3 with parameter values* $a = 0.5$, $\delta = 0.025$, $b = 0.1$, $\check{\rho} = 0.001$, $v = 0.04$, $\sigma = 0.2$, $\tau = 0.45$. *We find that there exist two equilibria, one with a low growth rate,* $g_1^{CDE} = 0.013$, *high rate of time preference,* $\bar{\rho}_1 = 0.101$, *low physical to human capital ratio,* $\bar{z}_1 = 4.57$, *and relatively high consumption to physical capital ratio,* $\bar{\omega}_1 = 0.22$, *and one with high growth rate,* $g_2^{CDE} = 0.073$, *low rate of time preference,* $\bar{\rho}_2 = 0.011$, *high physical to human capital ratio,* $\bar{z}_2 = 29.52$, *and low consumption to physical capital ratio,* $\bar{\omega}_2 = 0.01$.

2.3 Transitional dynamics and stability analysis

In this subsection we examine the relation between savings, the return on physical capital and growth, along with the complementarities of human capital on intertemporal utility. To this end, we analyze the transitional dynamics and local stability of the market economy, which are determined by the two-dimensional system of equations (13) and (14). In matrix notation we can write:

$$\begin{bmatrix} \dot{\omega} \\ \dot{z} \end{bmatrix} = \begin{bmatrix} \left(1 - \frac{\rho'(\cdot)\bar{z}}{\sigma}\right)\bar{\omega} & \left[(\frac{a}{\sigma} - 1)(1-\tau)(a-1)\bar{z}^{a-2} - \frac{1}{\sigma}\rho'(\cdot)\bar{\omega}\right]\bar{\omega} \\ -\bar{z} & \left[(1-\tau)(a-1)\bar{z}^{a-1} - va\tau\bar{z}^{a}\right] \end{bmatrix} \begin{bmatrix} \omega - \bar{\omega} \\ z - \bar{z} \end{bmatrix}$$

After some algebra, the determinant of the above system, J, is given by:

$$J = \bar{\omega}\bar{z}\left[\underbrace{-\frac{a}{\sigma}(1-\tau)(1-a)\bar{z}^{a-2} - va\tau\bar{z}^{a-1}}_{<0} - \frac{\rho'(\cdot)\bar{z}^{a-1}}{\sigma}[(1-\tau)a - v\tau(1+a)\bar{z}]\right]$$

The sign of J is ambiguous and depends on the parameters of the economy and the endogeneity of the rate of time preference. Under a constant rate of time preference, $\rho^{'}(\cdot) = 0$, the standard result of a unique growth rate and a steady-state ratio of physical to human capital stock that is saddle-path stable is obtained. However, when the rate of time preference is endogenous the local dynamics of the economy are nontrivial.

Proposition 2 *If at the steady state* $\bar{z} < \frac{a(1-\tau)}{v(1+a)\tau}$ *then* $\rho^{'}(\cdot) > 0$ *is a sufficient condition for local* *determinacy. If at the steady state* $\bar{z} > \frac{a(1-\tau)}{v(1+a)\tau}$ *then* $\rho'(\cdot) > 0$ *is a necessary condition for local indeterminacy.*

Proof. Follows straightforward from the sign of the determinant, J. ▪

Proposition 2 shows that for a sufficiently high steady-state ratio of physical to human capital stock, $\bar{z}$, the steady-state can be indeterminate when $\rho'(\cdot) > 0$. The intuition on how the endogeneity of time preference to the ratio of aggregate consumption to aggregate human capital may generate an intertemporal complementarity between savings and their return is as follows. Consider what happens if agents decide to increase savings. In the Ramsey–Cass–Koopmans model current consumption jumps downwards, capital accumulates at a faster rate and the economy achieves a higher equilibrium stock of capital. During the transition the rate of return on capital, r, falls and the growth rate, driven by $(r - \rho)$, falls under constant time discounting, thus acting as a stabilizing force to the initial increase in savings. This also happens here for low values of physical to human capital ratio, $\bar{z} < \frac{a(1-\tau)}{v(1+a)\tau}$, where the rate of return on capital falls and the subjective discount rate is an increasing function of z (see Proof of Proposition 1) preserving the stabilizing force towards the BGP. However, for sufficiently large values of $\bar{z} > \frac{a(1-\tau)}{v(1+a)\tau}$ the rate of time preference becomes a decreasing function of z and the standard counteracting force does not guarantee a unique path. The reason is that although the rate of return on capital falls, the growth rate of consumption, determined by $r - \rho(\cdot)$, can increase due to the fall in impatience. Thus, a continuum of equilibrium paths towards the steady-state can occur depending on the parameters of the model.

Due to the complexity for the computation of the determinant sign of the dynamic system we will numerically characterize the stability and dynamic properties using the assumptions of Example 1.

Example 2 *Consider a linear time preference function,* $\rho(\frac{C}{H}) = b * (\frac{C}{H}) + \check{\rho}$, *that satisfies Assumptions 1-3 with parameter values* $a = 0.5$, $\delta = 0.025$, $b = 0.1$, $\check{\rho} = 0.001$, $\sigma = 0.2$, $v = 0.04$, $\tau = 0.45$. *We can characterize the stability and dynamics of the system by computing its determinant and trace. We get that both steady-state values of* z *and* ω *are locally stable. However, the first is saddle-path and thus determinate, whereas the second is a node and thus indeterminate. In particular, in the first steady state we have* $z_1 = 4.57$, $\omega_1 = 0.228$, *and the determinant of the system is negative* ($J = -0.109$)*, which implies saddle-path stability. In the second steady state we* *get,* $z_2 = 29.52$*,* $\omega_2 = 0.003$*, and the determinant is positive (*$J = 0.004$*). The type of stability given by* $trace^2 - 4J = 0.0045 > 0$ *is a node and the trace is negative (*$trace = -0.146$*), which implies that the steady state is stable.*

Example 2 shows that when there are two stable BGPs, the one with the higher growth rate corresponds to the lower steady-state value of the consumption-to-capital ratio, higher physical-to-human capital ratio and is locally indeterminate, whereas the other one with the lower growth rate corresponds to the lower steady-state value of physical-to-human capital ratio and is locally determinate. Thus, consistent with the conditions of Propositions 1 and 2 we can have both local and global indeterminacy, which implies that given an initial condition it is not possible in the market economy to choose an initial value of consumption to either place the economy in the determinate BGP with the low growth rate or in the neighborhood of the indeterminate BGP with the high growth rate.[7] Our next task is to endogenize government policy and examine its role in leading the economy to a unique equilibrium path.

3 Time Preference and Ramsey Taxation

Previous studies of taxation and spending policies have assumed that a government is endowed with nondistortionary policy instruments (e.g. lump-sum taxes or transfers). In turn, the public finance literature has assumed that the government has a comprehensive mechanism (e.g. Pigouvian taxation) for fully internalizing any market failures from externalities. Appendix 2 shows that in our context the market economy cannot replicate the Pareto optimal allocation. In this section we endogenize fiscal policy and we examine the equilibrium selection mechanism of the governments' objective in the context of Ramsey taxation.

3.1 Ramsey allocation

In the current setup there exists a range of the initial endowments of the aggregate physical and human capital stocks in the CDE under which the economy will exhibit multiplicity for any tax rate. We examine here if, and how, the government's objective can impose restrictions and lead to an initial endowment allocation that solves the indeterminacy problem.

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[7] These results are robust to alternative parameter values. Notice that in the case of uniqueness, e.g. for a logarithmic utility function with $\sigma = 1$, the steady-state ratio of physical to human capital is saddle-path stable.

Definition 2 *Ramsey taxation is given under Definition 1 when (i) the government chooses the tax rate and aggregate allocations in order to maximize the welfare of the economy by taking into account the aggregate optimality conditions of the CDE, and (ii) the government budget constraint and the feasibility and technological conditions are met.*

The government seeks to maximize welfare of the economy subject to the outcome of the decentralized equilibrium summarized by (9)-(11). The Hamiltonian of this problem is given by:

$$\begin{aligned}\Lambda^R &= u(C)e^{-\Delta} + \frac{1}{\sigma}\tilde{\mu}_C C\left[a(1-\tau)\left(\frac{K}{H}\right)^{a-1} - \rho(\cdot) - \delta\right] + \\ &\quad \tilde{\mu}_K\left[(1-\tau)K^a\left(H\right)^{1-a} - C - \delta K\right] + \tilde{\mu}_H[v\tau K^a H^{1-a} - \delta H] + \tilde{\mu}_\rho[\rho(\cdot)]\end{aligned}$$

where $\tilde{\mu}_C$, $\tilde{\mu}_K$, $\tilde{\mu}_H$ are the dynamic multipliers associated with (9), (10), (11) respectively and $\Delta \equiv \int_0^t \rho(C_s, H_s)ds$.

The first-order conditions of the Ramsey problem include the constraints (9)-(11) and the optimality conditions with respect to C, H, K, τ:

$$\dot{\tilde{\mu}}_C = -\left[C^{-\sigma}\right]e^{-\Delta} - \frac{\tilde{\mu}_C}{\sigma}\left[a(1-\tau)\left(\frac{K}{H}\right)^{a-1} - \rho(\cdot) - \delta\right] + \left[\frac{\tilde{\mu}_C C}{\sigma} - \tilde{\mu}_\rho\right]\rho'(\cdot)\frac{1}{H} + \tilde{\mu}_K \tag{15}$$

$$\dot{\tilde{\mu}}_K = \frac{\tilde{\mu}_C C}{\sigma}\left[a(1-a)(1-\tau)\left(\frac{K}{H}\right)^{a-1}K^{-1}\right] - \tilde{\mu}_K\left[a(1-\tau)\left(\frac{K}{H}\right)^{a-1} - \delta\right] - \tilde{\mu}_H v\tau a\left(\frac{K}{H}\right)^{a-1} \tag{16}$$

$$\begin{aligned}\dot{\tilde{\mu}}_H &= -\frac{\tilde{\mu}_C C}{\sigma}\left[a(1-a)(1-\tau)\left(\frac{K}{H}\right)^{a}K^{-1} + \rho'(\cdot)\frac{C}{H^2}\right] \\ &\quad -\tilde{\mu}_K(1-a)(1-\tau)\left(\frac{K}{H}\right)^{a} - \tilde{\mu}_H\left[v(1-a)\tau\left(\frac{K}{H}\right)^{a} - \delta\right] + \tilde{\mu}_\rho\rho'(\cdot)\frac{C}{H^2}\end{aligned} \tag{17}$$

$$\frac{\tilde{\mu}_C aC}{\sigma}\left(\frac{K}{H}\right)^{a-1} + (\tilde{\mu}_K - v\tilde{\mu}_H)\left(\frac{K}{H}\right)^{a}H = 0 \tag{18}$$

$$\dot{\tilde{\mu}}_\rho = \frac{C^{1-\sigma}}{1-\sigma}e^{-\Delta} \tag{19}$$

$$C^{1-\sigma}e^{-\Delta} + \tilde{\mu}_C\dot{C} + \tilde{\mu}_K\dot{K} + \tilde{\mu}_H\dot{H} + \tilde{\mu}_\rho[\rho(\cdot)] = 0 \tag{20}$$

Equations (15)-(19), the optimality condition for the Hamiltonian $\lim_{t\to\infty}\Lambda^R = 0$ as given by (20), and equations (9), (10), (11) characterize the solution of the Ramsey problem. The methodology to derive the stationary Ramsey allocation is similar to that of the social planner's problem (see Appendix 2). Let us define $\mu_j \equiv \tilde{\mu}_j e^{\Delta(t)}$ where $j = C$, K, H, ρ. We can now transform the variables by defining $\omega \equiv \frac{C}{K}$, $z \equiv \frac{K}{H}$, $\psi \equiv \mu_C C$, $\phi \equiv \mu_K K$, $\chi \equiv \mu_H H$. Using the stationary variables, the dynamics of (15)-(20) and (9)-(11) are equivalent to the following dynamics:

$$\frac{\dot{\omega}}{\omega} = (\frac{a}{\sigma} - 1)(1-\tau)z^{a-1} + \omega - \frac{1}{\sigma}\rho(\omega z) - (\frac{1}{\sigma} - 1)\delta \tag{21}$$

$$\frac{\dot{z}}{z} = (1-\tau)z^{a-1} - \omega - v\tau z^{a} \tag{22}$$

$$-\frac{1}{\sigma}\psi a z^{a-1} - \phi z^{a-1} + v\chi z^{a} = 0 \tag{23}$$

$$\frac{\dot{\chi}}{\chi} = v\tau a z^{a} - \frac{(1-\tau)(1-a)z^{a-1}}{\chi}\left(\frac{a\psi}{\sigma} + \phi\right) + \rho(\omega z) - \frac{\rho'(\cdot)\omega z}{\chi}\left(\frac{\psi}{\sigma} - \mu_\rho\right) \tag{24}$$

$$\frac{\dot{\phi}}{\phi} = (1-\tau)(1-a)z^{a-1}\left(1 + \frac{a}{\sigma}\frac{\psi}{\phi}\right) - \omega - \frac{\chi}{\phi}v\tau a z^{a} + \rho(\omega z) \tag{25}$$

$$\frac{\dot{\psi}}{\psi} = -\frac{C^{1-\sigma}}{\psi} + \frac{\omega\phi}{\psi} + \rho(\omega z) + \rho'(\cdot)\omega z\left(\frac{1}{\sigma} - \frac{\mu_\rho}{\psi}\right) \tag{26}$$

$$\frac{\dot{\mu}_\rho}{\mu_\rho} = \frac{C^{1-\sigma}}{(1-\sigma)\mu_\rho} + \rho(\omega z) \tag{27}$$

$$C^{1-\sigma} = -(1-\tau)z^{a-1}\left(\frac{\psi a}{\sigma} + \phi\right) + \delta\left(\frac{\psi}{\sigma} + \phi + \chi\right) + \omega\phi - v\tau\chi z^{a} + \frac{\psi\rho(\omega z)}{\sigma} \tag{28}$$

In the long run, $\frac{\dot{\omega}}{\omega} = \frac{\dot{z}}{z} = \frac{\dot{\chi}}{\chi} = \frac{\dot{\phi}}{\phi} = \frac{\dot{\psi}}{\psi} = \frac{\dot{\tau}}{\tau} = 0$. Also, for $\frac{\dot{\mu}_\rho}{\mu_\rho}$ to follow a BGP, $\frac{C^{1-\sigma}}{(1-\sigma)\mu_\rho}$ has to be constant, which implies that $\frac{\dot{\mu}_\rho}{\mu_\rho} = (1-\sigma)\frac{\dot{C}}{C} = (1-\sigma)(\tau z^{a} - \delta)$. In turn, the long-run solution of the Ramsey problem is given by a system of 8 equations, (21)-(28) with 8 unknowns, $\tilde{z}, \tilde{\omega}, \tilde{\tau}, \tilde{\psi}, \tilde{\phi}, \tilde{\chi}, \tilde{\mu}_\rho, \tilde{C}$. Since the impatience function is endogenous to the ratio of consumption to human capital and the economy as described in the CDE accepts a BGP at which C, K and H grow at the same rate for any government policy, then along such a path the rate of time preference is constant over time, $\rho'(\cdot) = 0$ (Palivos et al., 1997). Also, to reduce the dimensionality of the problem we solve (23)-(27) which yields that the second-best tax rate, $\tilde{\tau}$, depends on the long-run solution of $\tilde{\omega}$ and $\tilde{z}$ which, in turn, can be obtained by equations (21)-(22). After some algebra, the long-run allocation of the Ramsey environment is characterized by the following system of equations:

$$\tilde{\tau} = (1-a) - \frac{\rho(\tilde{\omega}\tilde{z})\tilde{z}^{-a}}{v} \tag{29}$$

$$(\frac{a}{\sigma} - 1)(1-\tilde{\tau})\tilde{z}^{a-1} + \tilde{\omega} - \frac{1}{\sigma}\rho(\tilde{\omega}\tilde{z}) - (\frac{1}{\sigma} - 1)\delta = 0 \tag{30}$$

$$(1-\tilde{\tau})\tilde{z}^{a-1} - \tilde{\omega} - v\tilde{\tau}\tilde{z}^{a} = 0 \tag{31}$$

The system of equations (29)-(31) yield the Ramsey tax rate, $\tilde{\tau}$, and $\tilde{\omega}$ and $\tilde{z}$ as functions of the parameters. The following Proposition summarizes the properties of the equilibrium in the Ramsey environment.

Proposition 3 *The equilibrium tax rate and growth rate in the Ramsey environment are unique.*

Proof. See Appendix 3. ▪

Proposition 3 implies that fiscal policy in a second-best environment can act as an equilibrium selection device. In particular, Proposition 3 shows that, for any time preference function that satisfies Assumptions 1-3 and any parameter values in its assumed domain that generate multiplicity, Ramsey taxation leads to a unique competitive equilibrium allocation $(\tilde{z}, \tilde{\omega})$.

To analyze the behavior of the economy under the Ramsey allocation and highlight the economic intuition of our analytical results we present below numerical solutions.

Example 3 *Consider a linear time preference function, $\rho(\frac{C}{H}) = b * (\frac{C}{H}) + \check{\rho}$, that satisfies Assumptions 1-3 with parameter values $a = 0.5$, $\delta = 0.025$, $b = 0.1$, $\check{\rho} = 0.001$, $v = 0.04$, $\sigma = 0.2$. Under the Ramsey allocation we find a unique equilibrium growth rate given by $g^R = 0.073$ corresponding to a tax rate given by $\tilde{\tau} = 0.449$.*

Example 3 shows that for the parameter values under which the CDE exhibits multiple equilibria (see Example 1), the government attains a unique equilibrium by implementing the appropriate allocation and restrictions in the Ramsey environment. Intuitively, this happens because the government uses the allocation of aggregate endowments and an associated tax rate to select a stable equilibrium regime and attain welfare maximization. Formally, this is accomplished through the state-dependent taxation rule, given by equation (29).

Concerning standard literature, global and local indeterminacy at the CDE implies that the rational expectations equilibria involve random variables, which are unrelated to the economy's fundamentals and are driven by individual beliefs (Benhabib and Farmer, 1994; Benhabib and Perli, 1994). Although indeterminacies have been widely studied in the literature, far less is known on mechanisms in directing the economy towards a desired equilibrium. In models with a continuum of equilibria arising from the presence of animal spirits, learning can act as a selection device for choosing the rational expectations equilibrium that we can expect to observe in practice (Evans et al., 1998; Evans and Honkapohja, 1999). Ennis and Keister (2005) have presented a framework in which search frictions create a coordination problem that generates multiple Pareto-ranked equilibria and show how the desired equilibrium can be chosen using a selection mechanism based on risk dominance. Antinolfi et al. (2007) analyze multiplicity in a model with heterogeneous agents and intertemporal complementarities between dated debt limits, which exhibits two Pareto-ranked equilibria, and show how active monetary policy can force the economy onto the optimal path. Here the government selects a unique equilibrium regime through the endogenous allocation of endowments and the choice of a feasible tax rate in a dynamic second-best environment. Equilibrium selection is feasible since intertemporal complementarities, which fuel multiplicity and are external to the agents in the CDE, are internalized under Ramsey taxation. Hence, the government obtains information through the agents' reaction function and consequently imposes restrictions on the tax rate and endowment allocation through commitment in order to drive the CDE to a unique equilibrium.[8]

3.2 Equilibrium selection with Ramsey taxation: An example

In this subsection we illustrate numerically the role of Ramsey taxation as an equilibrium selection device. As a first step, we provide the CDE outcome (exogenously set tax rate) for values of intertemporal elasticity of substitution under which multiplicity arises. Next, we show that for this parameter range the Ramsey allocation leads the economy to a unique competitive equilibrium.[9]

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[8] The transitional dynamics of the Ramsey problem can be obtained by investigating the system of differential equations (21)-(22) after applying the solution for the optimal tax rate given by (29). The conditions for local stability and uniqueness for the transitional dynamics of the Ramsey problem are available upon request.

[9] Notice that the results presented in this subsection could be obtained for any model parameter. We focus on the intertemporal elasticity of substitution because it stresses the intertemporal responsiveness of consumption to changes marginal utility and thus yields more intuitive results when the rate of time preference is endogenous to consumption. Alternatively, we can use the (endogenously chosen) unique Ramsey tax rate as an exogenously set parameter to obtain the CDE multiple equilibrium outcome.

In particular, the upper panel of Table 1 provides a sensitivity analysis of the result on multiplicity by showing the response of the CDE allocation to changes in the values of σ under which equilibrium indeterminacy is expected to occur according to the parameter values and functional forms of Example 1. As can be easily verified, the condition for multiplicity holds for all parametric combinations of Table 1. An increasing σ (declining elasticity of substitution of consumption over time) implies that agents are more averse in substituting current consumption for future one. As a result, when the economy is in the "low-growth" equilibrium current consumption is higher, which in turn implies a higher discount rate and less savings leading to lower capital accumulation and a lower value of the physical to human capital ratio, z. Output will be lower, which implies that for any tax rate there will be less room for human capital financing, and thus the effects stemming from the rise of consumption on the discount rate are reinforced. On the other hand, when the economy is in the "high growth" equilibrium the rise in σ, which raises current consumption, induces a higher z and a lower consumption to physical capital ratio, ω, as long as σ is sufficiently low. Output increases and human capital rises leading to a lower steady-state consumption to human capital ratio mirrored in the fall of the discount rate that in turn induces savings and higher steady-state growth. This happens because in the "high-growth" equilibrium the dynamics of the economy are sufficient to accommodate the initial increase in current consumption with an increase in human capital expenditures through the higher tax base (growth rate), thus leading to opposite changes in the variables of the economy. In contrast, in the lower panel of Table 1 where σ is sufficiently high and exceeds the elasticity of physical capital in the production function, any rise in σ simply leads to lower capital accumulation and output, which reduces human capital and raises the discount rate, thus precluding any other effects that fuel multiplicity.

In turn, Table 2 presents the response of the Ramsey allocation to changes in the value of σ under which the CDE is characterized by multiplicity. As can be readily verified, the Ramsey environment resolves indeterminacy and leads the economy to the "high-growth" regime. In particular, Table 2 shows that the tax rate and the growth rate decrease with the intertemporal elasticity of substitution, while the ratio of physical to human capital stock and the subjective discounting are decreasing. Intuitively, following an increase in σ agents are more averse in substituting current consumption for future one, which in turn implies a higher discount rate and less savings leading to lower capital accumulation and a lower value of the physical to human capital ratio, z. At the same time a lower

z decreases the marginal cost of public funds by reducing the distorting effect of taxation on private savings, lowers the tax base and results, at the BGP, in an increase in the tax rate. The endogenous change in the tax rate increases public expenditures and human capital, lowers the rate of time preference and thus results in an increase of the growth rate. Note that, in accordance with our analysis of local stability, the allocation of human and physical capital endowments belong to that path where the declining physical to human capital ratio is accompanied by a decreasing rate of time preference, thus providing a reinforcing growth effect through the Euler equation and in turn capturing the desired regime. Hence, the endogenous change in the tax rate maximizes welfare by internalizing private decisions. The appropriate allocation of aggregate endowments places then the economy onto the desired regime. These channels will be studied in more detail below when we analyze the growth-maximizing fiscal policy allocation and the associate comparative statics, which has been extensively studied in the context of policy rules in the public finance literature.

4 Growth-Maximizing Fiscal Policy and Comparative Statics

In this section we analyze growth-maximizing fiscal policy rules. Modern growth theory has shown particular interest in growth-enhancing policies, as the understanding of the forces of economic growth is crucial in order to identify the relative merits and synergies of government interventions. Moreover, the growth rate is usually the main measurable objective of the government. Although earlier papers, like Barro (1990), have mostly considered welfare and growth-maximizing policies under a unified perspective, subsequent studies have emphasized the role of growth maximization as an independent policy target.[10]

Definition 3 *A growth-maximizing (GM) allocation in the competitive equilibrium of the aggregate economy is given under Definition 1 when (i) the government chooses the tax rate and aggregate allocations in order to maximize the growth rate of the economy by taking into account the aggregate optimality conditions of the CDE, and (ii) the government budget constraint and the feasibility and technological conditions are met.*

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[10] See Economides et al. (2007) and Dioikitopoulos and Kalyvitis (2010) for a similar approach.

The government seeks to maximize the growth rate of the economy, g, given by:

$$\max_{z,\tau} g = v\tau z^a - \delta$$

subject to the equilibrium CDE response summarized by $(\frac{a}{\sigma})(1-\tau)z^{a-1} - v\tau z^a - \frac{1}{\sigma}\rho(\omega(z)z) - (\frac{1}{\sigma} - 1)\delta = 0$ and $\omega(z) = (1-\tau)z^{a-1} - v\tau z^a$.

The first-order conditions with respect to z and τ are:

$$av\hat{\tau}\hat{z}^{a-1} + (\frac{a}{\sigma})(a-1)\hat{\lambda}(1-\hat{\tau})\hat{z}^{a-2} - \hat{\lambda}v\hat{\tau}a\hat{z}^{a-1} - \frac{1}{\sigma}\hat{\lambda}\left[a(1-\hat{\tau})\hat{z}^{a-1} - v(a+1)\hat{\tau}\hat{z}^a\right]\rho'(\cdot) = 0 \quad (32)$$

$$v\hat{z}^a - (\frac{a}{\sigma})\hat{\lambda}\hat{z}^{a-1} - v\hat{\lambda}\hat{z}^a + \frac{1}{\sigma}\hat{\lambda}\left[\hat{z}^a + v\hat{z}^{a+1}\right]\rho'(\cdot) = 0 \quad (33)$$

where $\hat{\lambda}$ is the associated Lagrange multiplier, and $\hat{z}$ and $\hat{\tau}$ are the GM values of z and τ respectively. Solving (33) for $\hat{\lambda}$ and substituting in (32) we can obtain the following system of equations that characterize the GM policy rules:

$$\hat{\tau} = \frac{a(1-a+\rho'(\cdot)\hat{z})}{a+v\rho'(\cdot)\hat{z}^2} > 0 \quad (34)$$

$$(\frac{a}{\sigma})(1-\hat{\tau})\hat{z}^{a-1} - v\hat{\tau}\hat{z}^a - \frac{1}{\sigma}\rho(\omega(\hat{z})\hat{z}) - (\frac{1}{\sigma} - 1)\delta = 0 \quad (35)$$

Equation (34) yields the GM tax rate, $\hat{\tau}$. Notice that when the rate of time preference is constant ($\rho'(\cdot) = 0$) the government has to implement a marginal tax rate that is equal to the elasticity of publicly provided human capital in the production function, $\tau = (1-a)$, as in Barro (1990), Futagami et al. (1993) and Glomm and Ravikumar (1997). However, under endogenous time preference ($\rho'(\cdot) \neq 0$) the GM tax rate can be lower or higher than the elasticity of human capital in the production function since the tax policy also depends on demand-driven parameters.

To highlight these points we provide some numerical examples for a range of parameter values to check equilibrium selection under the GM allocation and how the comparative statics evolve. For comparison purposes we illustrate below these analytical results using the parameter values of Example 1 for which the CDE exhibits multiplicity.

Example 4 *Consider a linear time preference function,* $\rho(\frac{C}{H}) = b * (\frac{C}{H}) + \check{\rho}$, *that satisfies Assumptions 1-3 with parameter values* $a = 0.5$, $\delta = 0.025$, $b = 0.1$, $\check{\rho} = 0.001$, $v = 0.04$, $\sigma = 0.2$. *The GM* *tax rate is given by* $\hat{\tau} = 0.473$ *with respective growth rate* $g^{GM} = 0.073$*, physical to human capital ratio* $\hat{z} = 26.67$*, rate of time preference* $\hat{\rho} = 0.012$*, consumption to physical capital ratio* $\hat{\omega} = 0.004$ *and consumption to human capital ratio* $\frac{\hat{C}}{H} = 0.107$*.*

Example 4 shows that under the parameter values that produce multiplicity in the CDE, the GM allocation can act as an equilibrium selection device and impose the allocation restrictions and the tax rate that guarantee a unique equilibrium growth. Notice that in the GM allocation the "high-growth" BGP is selected, a result that is consistent with the government's objective.[11]

In Table 3 we examine the sensitivity of equilibrium selection to changes in key parameters and the implications for the tax rate in comparison with the standard Barro (1990) taxation rule. As can be readily seen, the equilibrium selection property for the GM allocation is robust to changes in σ, for which there is multiplicity of equilibria in the CDE. Table 3 then shows the response of the tax rate to changes in σ and the associate change in the comparative statics. For $0.1 < \sigma < 0.3$ the GM tax rate is lower than the one dictated by the Barro (1990) taxation rule (0.5), whereas for $0.4 < \sigma < 1$ the corresponding GM tax rate is higher. According to Table 3 for sufficiently low levels of σ, $0.1 < \sigma < 0.3$, an increase in the intertemporal elasticity of substitution increases the GM tax rate and the growth rate of the economy, and results in a lower physical to human capital ratio, lower consumption to human capital ratio and lower rate of time preference. Intuitively, following an increase in σ agents are more averse in substituting current consumption for future one, which in turn implies a higher discount rate and less savings leading to lower capital accumulation and a lower value of the physical to human capital ratio, z. This lowers the marginal cost of public funds and the tax base of the economy, thus resulting in an endogenous increase in the tax rate at the BGP that activates two opposing channels. On the one hand, the increase in the tax rate lowers capital accumulation, increases consumption (depending on the level of σ), the rate of time preference and in turn decreases the growth rate. On the other hand, an increase in the tax rate increases public expenditures and human capital, lowers the rate of time preference and raises the growth rate of the economy. Hence, for a low level of σ the effect of the endogenous change in the tax rate on consumption and the implied positive effect on the rate of time preference are low and the second channel dominates, resulting in a lower $\frac{C}{H}$ that in turn lowers time preference and raises

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[11] For the parameter values used in Example 4 and the growth maximizing tax rate $\hat{\tau} = .473$, the CDE gives two equilibrium growth rates $\bar{g}_1 = 0.0154$ and $\bar{g}_2 = 0.07277$.

growth (see the effects of σ on $\hat{g}$, $\frac{\hat{C}}{H}$, $\hat{\rho}$ for sufficiently low values of σ, $0.1 < \sigma < 0.3$). In this case, a higher growth rate forms a higher tax base, the marginal benefit of public funds increases and the growth-maximizing tax rate has to be lower than the elasticity of human capital in the production function. In contrast, for sufficiently high levels of σ, $0.4 < \sigma < 1$, the first channel dominates, resulting in a lower growth rate, a lower tax base and marginal benefit of public funds leading to a tax rate that is higher than the Barro (1990) taxation rule.

An immediate implication of the results presented in Table 3 is that the changes of the endogenous variables depend qualitatively on the local stability properties of the CDE. Recall that for $0.1 < \sigma < 0.3$ the CDE is multiple, while the GM allocation selects the "high-growth" equilibrium. For $0.1 < \sigma < 0.3$ the "high-growth" BGP is locally indeterminate, whereas for $0.4 < \sigma < 1$ the unique CDE equilibrium is locally determinate. In these two parametric ranges the responses of the variables are in the opposite direction, an outcome that is consistent with Samuelson's correspondence principle, which states that the comparative statics depend crucially on the local properties of the equilibrium. In particular, the results derived in Table 3 indicate, first, that an increase in σ for $0.1 < \sigma < 0.3$ raises the tax rate and the growth rate, whereas an increase in σ for $0.4 < \sigma < 1$ lowers the tax rate and growth. The effect of an increasing low (high) σ on the growth-maximizing tax rate is lower (higher) than the Barro (1990) tax rate.

The preceding discussion highlights the role of the slope of impatience function, b, in the qualitative response of the economy. Table 4 shows the response of the economy to changes in b. The upper panel of Table 4 indicates that if we set parameter values where the CDE is characterized by multiple equilibria, the equilibrium selection property of the GM allocation is not affected by changes in b. In turn, the lower panel of Table 4 checks the response of the Barro (1990) taxation rule to changes in b and, in conjunction with the upper panel, provides a picture of the response of the endogenous allocation of the GM allocation problem to changes in the slope of impatience function. In particular, an increase in b leads to an increase in the tax rate and to a decrease in the physical to human capital stock ratio, whereas the effects on the rate of time preference and the growth rate are ambiguous and depend on the level of the intertemporal elasticity of substitution. Intuitively, an increase in the slope of the impatience function increases ceteris paribus the rate of time preference, which lowers savings and capital accumulation and in turn decreases the physical to human capital ratio. Also, by the Euler equation an increase in the rate of time preference lowers the growth rate and the tax base of the economy, and generates an endogenous increase in the tax rate to finance public expenditures at the BGP. In turn, the endogenous increase in the tax rate activates the previously analyzed mechanism. For a sufficiently high level of the intertemporal elasticity of substitution (e.g. $\sigma = 1$), the rise in the tax rate increases consumption more than human capital expenditures and reinforces the initial increase in the rate of time preference leading to an additional decrease in the growth rate of the economy. In contrast, when σ is sufficiently low (e.g. $\sigma = 0.3$) an increase in the tax rate increases human capital expenditures leading to an increase in the growth rate of the economy which counteracts the initial decrease. Also, in the latter case the increase in the tax rate lowers the consumption to human capital ratio, since human capital expenditures increase more than consumption for low σ, leading to lower rate of time preference and counteracting the first order increase. As summarized in Table 5, for low values of σ the response of the rate of time preference to the decrease in the consumption to human capital ratio, $\frac{C}{H}$, is high and dominates the initial exogenous increase of the rate of time preference caused by b, whereas for high values of σ the initial increase in ρ dominates its endogenous decrease driven by $\frac{C}{H}$.

5 Concluding Remarks

This paper studied the macroeconomic implications of the endogeneity of time preference to aggregate human capital provided by the public sector. We derived the long-run behavior of the economy and analyzed the impact of fiscal policy. The main findings are that multiple BGPs emerge in the decentralized economy and that second-best (Ramsey and growth-maximizing) taxation can act as an equilibrium selection device in order to lead the economy to a desired BGP. In addition, we also challenged the standard fiscal policy rules of the public finance literature and showed that their qualitative properties depend on demand-side parameters through intertemporal preferences. Therefore, to the extent that government policies affect time discounting by enhancing education and other "future-oriented" policies, our findings suggest a channel for the impact of fiscal policy on long-run growth that has been left unnoticed in existing studies and warrants further analysis.

Appendix 1: Proof of Proposition 1

The method will be to separate function $\Phi(z)$ in two functions and find their intersection to solve it. We define $\Gamma(z) \equiv (\frac{a}{\sigma})(1-\tau)(z)^{a-1} - v(z)^a\tau - (\frac{1}{\sigma}-1)\delta$ and $\Lambda(z) \equiv \frac{1}{\sigma}\rho(z \cdot \omega(z))$. Both $\Gamma(z)$ and $\Lambda(z)$ are continuous in z. In order for $\omega(z) > 0$ to hold we must have $z < \frac{1-\tau}{v\tau}$.

Equation $\Gamma(z)$ has the following properties:

1. $\lim\limits_{z \to 0} \Gamma(z) = +\infty$, $\lim\limits_{z \to \frac{1-\tau}{v\tau}} \Gamma(z) = (\frac{a}{\sigma}-1)(1-\tau)^a\tau^{1-a}v^{1-a} - (\frac{1}{\sigma}-1)\delta$.

2. $\frac{\partial\Gamma(z)}{\partial z} < 0$, $\frac{\partial^2\Gamma(z)}{\partial z^2} > 0$.

From the properties of $\Gamma(z)$ is follows that it is a strictly decreasing and convex function in its domain, starts from $+\infty$ and ends at $(\frac{a}{\sigma}-1)(1-\tau)^a\tau^{1-a}v^{1-a} - (\frac{1}{\sigma}-1)\delta$.

Equation $\Lambda(z)$ has the following properties:

1. $\lim\limits_{z \to 0} \Lambda(z) = \frac{1}{\sigma}\rho(0) = \frac{1}{\sigma}\breve{\rho}$, $\lim\limits_{z \to \frac{1-\tau}{\tau}} \Lambda(z) = \frac{1}{\sigma}\rho(0) = \frac{1}{\sigma}\breve{\rho}$.

2. $\frac{\partial\Lambda(z)}{\partial z} = \frac{1}{\sigma}\rho'(.)\left[a(1-\tau)z^{a-1} - v\tau(1+a)z^a\right]$. We have $\frac{\partial\Lambda(z)}{\partial z} > 0$ for $a(1-\tau)z^{a-1} - v(1+a)\tau z^a > 0 \Rightarrow z < \frac{a(1-\tau)}{v(1+a)\tau}$ and $\frac{\partial\Lambda(z)}{\partial z} < 0$ for $z > \frac{a(1-\tau)}{v(1+a)\tau}$. Thus, $\Lambda(z)$ has a maximum at $z = \frac{a(1-\tau)}{v(1+a)\tau}$.

From the properties of $\Lambda(z)$ it follows that it is an inverse U-shaped curve starting from $\frac{1}{\sigma}\breve{\rho}$ and ending at $\frac{1}{\sigma}\breve{\rho}$.

Assuming equilibrium existence, from the properties of $\Lambda(z)$ and $\Gamma(z)$ it follows that there exist one or two positive equilibrium growth rates. For low values of z, since $+\infty > \frac{1}{\sigma}\breve{\rho}$ we get that $\Gamma(z)$ lies above $\Lambda(z)$. Also, for the upper bound value of z, $\Gamma(z) = (\frac{a}{\sigma}-1)(1-\tau)^a\tau^{1-a}v^{1-a} - (\frac{1}{\sigma}-1)\delta$ and $\Lambda(z) = \frac{1}{\sigma}\breve{\rho}$. Since both functions are continuous, if $(\frac{a}{\sigma}-1)(1-\tau)^a\tau^{1-a}v^{1-a} - (\frac{1}{\sigma}-1)\delta < \frac{1}{\sigma}\breve{\rho}$, which means that $\Gamma(z)$ starts above and ends below $\Lambda(z)$ implying that $\Gamma(z)$ will cross $\Lambda(z)$ once and there will exist a unique equilibrium growth rate. Thus, $(\frac{a}{\sigma}-1)(1-\tau)^a\tau^{1-a}v^{1-a} - (\frac{1}{\sigma}-1)\delta < \frac{1}{\sigma}\breve{\rho}$ is a sufficient parametric condition for a unique equilibrium growth rate.

If $(\frac{a}{\sigma}-1)(1-\tau)^a\tau^{1-a}v^{1-a} - (\frac{1}{\sigma}-1)\delta \geq \frac{1}{\sigma}\breve{\rho}$ then there can exist two equilibrium growth rates because $\Lambda(z)$ is an inverse U-shaped curve while $\Gamma(z)$ strictly monotone and decreasing, so $\Gamma(z)$ can cross $\Lambda(z)$ at most two times. Thus, $(\frac{a}{\sigma}-1)(1-\tau)^a\tau^{1-a}v^{1-a} - (\frac{1}{\sigma}-1)\delta \geq \frac{1}{\sigma}\breve{\rho}$ is a necessary parametric condition for multiplicity.

In order for this condition to be sufficient we need to find the parametric condition under which $\Lambda(z)$ cannot be tangent to $\Gamma(z)$. If they are tangent, since $\Gamma(z)$ is always decreasing, this has to be at the region where $\Lambda(z)$ is decreasing, i.e. $z > \frac{a(1-\tau)}{v(1+a)\tau}$. In other words, we need to prove that there cannot be an intersection of the first derivatives of $\Lambda(z)$ and $\Gamma(z)$, $\frac{\partial\Lambda(z)}{\partial z} \neq \frac{\partial\Gamma(z)}{\partial z}$, for $z > \frac{a(1-\tau)}{v(1+a)\tau}$.

Equation $\frac{\partial\Lambda(z)}{\partial z}$ has the following properties: $\lim\limits_{z \to \frac{a(1-\tau)}{v(1+a)\tau}} \frac{\partial\Lambda(z)}{\partial z} = 0$, $\lim\limits_{z \to \frac{1-\tau}{v\tau}} \frac{\partial\Lambda(z)}{\partial z} = -\frac{\tau}{\sigma}\rho'(0)(\frac{1-\tau}{v\tau})^a$, $\frac{\partial^2\Lambda(z)}{\partial z^2} = \frac{1}{\sigma}\rho''(.)\left[a(1-\tau)z^{a-1} - v\tau(1+a)z^a\right]^2 + \frac{1}{\sigma}\rho'(.)\left[a(a-1)(1-\tau)z^{a-2} - v\tau a(1+a)z^{a-1}\right] < 0$ for $\rho''(.) \leq 0$. Thus, for $\rho''(.) \leq 0$, $\frac{\partial\Lambda(z)}{\partial z}$ is a monotonically decreasing function starting from 0 and ending at $-\frac{\tau}{\sigma}\rho'(0)(\frac{1-\tau}{v\tau})^a$.

Equation $\frac{\partial\Gamma(z)}{\partial z}$ has the following properties: $\lim\limits_{z \to \frac{a(1-\tau)}{v(1+a)\tau}} \frac{\partial\Gamma(z)}{\partial z} = \tau(\frac{a(1-\tau)}{v(1+a)\tau})^{a-1}v[(a-1)(\frac{(1+a)}{\sigma}) - a]$,

$\lim\limits_{z\longrightarrow\frac{1-\tau}{v\tau}} \frac{\partial\Gamma(z)}{\partial z} = a\tau(\frac{1-\tau}{v\tau})^{a-1}v[(\frac{a-1}{\sigma})-1]$, $\frac{\partial^2\Gamma(z)}{\partial z^2} = a(a-1)z^{a-2}[(\frac{a-2}{\sigma})(\frac{1-\tau}{z})-\tau] > 0$. Thus, $\frac{\partial\Gamma(z)}{\partial z}$ is an increasing function starting from $\tau(\frac{a(1-\tau)}{(1+a)\tau})^{a-1}v[(a-1)(\frac{1+a}{\sigma})-a]$ and ending at $a\tau(\frac{1-\tau}{v\tau})^{a-1}v[(\frac{a-1}{\sigma})-1]$.

Then, since $\frac{\partial\Lambda(z)}{\partial z}$ starts above $\frac{\partial\Gamma(z)}{\partial z}$, $\lim\limits_{z\longrightarrow\frac{a(1-\tau)}{(1+a)\tau}} \frac{\partial\Lambda(z)}{\partial z} > \lim\limits_{z\longrightarrow\frac{a(1-\tau)}{(1+a)\tau}} \frac{\partial\Gamma(z)}{\partial z}$, and both functions are monotone, then, a sufficient condition for non-intersection is that $\frac{\partial\Lambda(z)}{\partial z}$ ends above $\frac{\partial\Gamma(z)}{\partial z}$, that is $\lim\limits_{z\longrightarrow\frac{1-\tau}{v\tau}} \frac{\partial\Lambda(z)}{\partial z} > \lim\limits_{z\longrightarrow\frac{1-\tau}{v\tau}} \frac{\partial\Gamma(z)}{\partial z}$. This happens if $-\frac{\tau}{\sigma}\rho'(0)(\frac{1-\tau}{v\tau})^a > a\tau(\frac{1-\tau}{v\tau})^{a-1}v(\frac{a-1}{\sigma}-1) \Rightarrow -\frac{1}{\sigma}\rho'(0) > (\frac{va\tau}{1-\tau})v(\frac{a-1}{\sigma}-1)$. Thus, if $\rho''(.) \leq 0$ and $-\frac{1}{\sigma}\rho'(0) > (\frac{a\tau v^2}{1-\tau})(\frac{a-1}{\sigma}-1)$, condition $(\frac{a}{\sigma}-1)(1-\tau)^a\tau^{1-a}v^{1-a}-(\frac{1}{\sigma}-1)\delta \geq \frac{1}{\sigma}\check{\rho}$ is sufficient for the presence of two positive equilibrium growth rates.■

Appendix 2: Endogenous Impatience and First-Best Fiscal Policy

In this model the social planner (SP) seeks to maximize aggregate utility subject to the production technology, the aggregate budget constraint and the physical and the human capital accumulation technology.

Definition A1 *Let a path $\{K(t), H(t), \lambda_i(t)\}$ for $t \geq 0$ be the solution to the social planner problem, where $\lambda_i(t)$ are the costate variables associated with the constraints. We call it a BGP in the SP problem if the growth rates of these variables, g_K^{SP}, g_H^{SP}, $g_{\lambda_i}^{SP}$, are constant over time.*

Formally, the SP problem is to maximize aggregate discounted utility given by:

$$\int_0^\infty u(C)\exp\left[-\Delta(t)\right]dt \tag{A.1}$$

$$\dot{K} = K^aH^{1-a} - C - H^E - \delta K \tag{A.2}$$

$$\dot{\Delta} = \rho(\cdot) \tag{A.3}$$

and the human capital accumulation equation (6) and $K(0) > 0$, $H(0) > 0$, where $\Delta = \int_0^t \rho(C_s, H_s)ds$. The present value Hamiltonian of the problem is given by:

$$\Lambda^{SP} = \frac{C^{1-\sigma}}{1-\sigma}e^{-\Delta} + \tilde{\lambda}_K\left[K^aH^{1-a} - C - H^E - \delta K\right] + \tilde{\lambda}_H\left[vH^E - \delta H\right] + \tilde{\lambda}_\rho[\rho(\cdot)]$$

The first-order conditions are given by:

$$C^{-\sigma}e^{-\Delta} - \tilde{\lambda}_K + \tilde{\lambda}_\rho\rho'(\cdot)\frac{1}{H} = 0 \tag{A.4}$$

$$\tilde{\lambda}_K = v\tilde{\lambda}_H \tag{A.5}$$

$$-\frac{\dot{\tilde{\lambda}}_K}{\tilde{\lambda}_K} = a\left(\frac{K}{H}\right)^{a-1} - \delta \tag{A.6}$$

$$\tilde{\lambda}_K\left[(1-a)\left(\frac{K}{H}\right)^a\right]-\tilde{\lambda}_H\delta-\tilde{\lambda}_\rho\rho'(\cdot)\frac{C}{H^2}=-\dot{\tilde{\lambda}}_H \tag{A.7}$$

$$\dot{\tilde{\lambda}}_\rho=\frac{C^{1-\sigma}}{1-\sigma}e^{-\Delta} \tag{A.8}$$

$$\lim_{t\to\infty}\Lambda^{SP}(t)=0 \tag{A.9}$$

Defining $\lambda_i \equiv \tilde{\lambda}_i e^{\Delta(t)}$ where $i = K,\ H,\ \rho$ and using (A.5) and (A.3) it follows that

$$-\frac{\dot{\lambda}_K}{\lambda_K}=a\left(\frac{K}{H}\right)^{a-1}-\delta-\rho(\cdot) \tag{A.10}$$

Let g_λ denote the common growth rate of λ_K and λ_H. From (A.10) we have that $\rho(\cdot) = g_\lambda + a\left(\frac{K}{H}\right)^{a-1} - \delta$, which for equal growth rates of K and H implies a constant rate of time preference at the BGP (Palivos et al., 1997).

Result A1 *Under Assumptions 1-3 the rate of time preference in the SP economy is constant over time at the BGP and fiscal policy cannot replicate the first-best environment.*

Under Result A1 we have that (A.4) is given by:

$$C^{-\sigma}-\lambda_K=0 \tag{A.11}$$

Combined with (A.10) the social planner growth rate of consumption, g_C^{SP}, is:

$$g_C^{SP}=\frac{1}{\sigma}\left[a\left(\frac{K}{H}\right)^{a-1}-\delta-\rho(\cdot)\right] \tag{A.12}$$

Along the optimal trajectory the Hamiltonian is independent of time and, together with the transversality condition (A.9), we have that $\Lambda^{SP} = 0$, which implies that:

$$\lambda_\rho=-\frac{1}{\rho(\cdot)}\left[\frac{C^{1-\sigma}}{1-\sigma}+\lambda_K\left[K^aH^{1-a}-C-H^E-\delta K\right]+\lambda_H\left[vH^E-\delta_hH\right]\right] \tag{A.13}$$

From (A.3), (A.7) and the definitions of the costate variables we can get that:

$$-\frac{\dot{\lambda}_H}{\lambda_H}=v(1-a)\left(\frac{K}{H}\right)^a-\delta-\rho(\cdot)+\frac{\lambda_\rho}{\lambda_H}\rho'(\cdot)\frac{C}{H^2} \tag{A.14}$$

Combining (A.5), (A.10), (A.13), and (A.14), yields under Proposition 3 the costate variables λ_K, λ_H, λ_ρ, as well as the physical to human capital stock in the socially planned economy at the BGP as $\frac{K}{H} = \frac{a}{v(1-a)}$.

Suppose that there exists $\tau > 0$ such that the CDE replicates the allocations and thus the growth rates of the SP, i.e. that is $\left(\frac{K}{H}\right)^{CDE} = \left(\frac{K}{H}\right)^{SP} = \frac{a}{v(1-a)}$, $\left(\frac{C}{H}\right)^{CDE} = \left(\frac{C}{H}\right)^{SP}$, and $g_C^{CDE} = g_C^{SP}$. The growth rate under the SP and the CDE will be given respectively by $g_C^{SP} = \frac{1}{\sigma}\left[a\left(\left(\frac{K}{H}\right)^{SP}\right)^{a-1} - \delta - \rho\left(\cdot\right)^{SP}\right]$ and $g_C^{CDE} \equiv \frac{\dot{C}}{C} = \frac{1}{\sigma}\left[(1-\tau)a\left(\left(\frac{K}{H}\right)^{CDE}\right)^{a-1} - \delta - \rho\left(\cdot\right)^{SP}\right]$. The CDE can replicate the social planner solution, $g_C^{SP} = g_C^{CDE}$, only for $\tau = 0$. ■

The market economy cannot replicate the SP solution that requires to allocate initial endowments so that $\frac{K_0}{H_0} = \frac{a}{v(1-a)}$ and finance human capital expenditures with a lump-sum tax.

Appendix 3: Proof of Proposition 3

Equations (29)-(31) form a system of 3 equations with three positive unknowns, $\tilde{z}$, $\tilde{\omega}$ and $\tilde{\tau}$ that give the solution of the Ramsey problem. To establish the uniqueness of the solution we solve (29) for $\rho(\tilde{\omega}\tilde{z}) > 0$, which holds for $\tilde{\tau} < 1-a$, and (31) for $\tilde{\omega}$, and substitute in (30) to obtain: $\tilde{\tau} = (1-a) - \frac{\rho(\tilde{\omega}\tilde{z})\tilde{z}^{-a}}{v}$

$$V(\tilde{z},\tilde{\tau}) \equiv (\frac{a}{\sigma})(1-\tilde{\tau})(\tilde{z})^{a-1} - v(\tilde{z})^{a}\tilde{\tau} - (\frac{1}{\sigma}-1)\delta - \frac{v}{\sigma}\frac{(1-a)-\tilde{\tau}}{\tilde{z}^{-a}} = 0$$

Since we have used all three equilibrium equations to obtain the last equation, it is sufficient to characterize the existence and uniqueness of equilibrium bundle $(\tilde{z},\tilde{\tau})$ if there exists a bundle $(\tilde{z}^R,\tilde{\tau}^R)$ such that $V(\tilde{z}^R,\tilde{\tau}^R) = 0$ for any $\tilde{\tau} < 1-a$ and $\tilde{z}$ that satisfies the interior solution of the problem $(\tilde{z} > 0, \tilde{\omega} > 0, \tilde{\tau} > 0, \rho > 0)$. The method will be to separate function $V(\tilde{z},\tilde{\tau})$ in two functions and find their intersection to solve it. We define $\Gamma^R(\tilde{z},\tilde{\tau}) \equiv (\frac{a}{\sigma})(1-\tilde{\tau})(\tilde{z})^{a-1} - v(\tilde{z})^{a}\tilde{\tau} - (\frac{1}{\sigma}-1)\delta$ and $\Lambda^R(\tilde{z},\tilde{\tau}) \equiv \frac{v}{\sigma}(\frac{(1-a)-\tilde{\tau}}{\tilde{z}^{-a}})$. Both $\Gamma^R(\tilde{z},\tilde{\tau})$ and $\Lambda^R(\tilde{z},\tilde{\tau})$ are continuous in $\tilde{z}$ and $\tilde{\tau}$. Also, from (31) in order for $\tilde{\omega}(\tilde{z}) > 0$ to hold we must have $\tilde{z} < \frac{1-\tilde{\tau}}{v\tilde{\tau}}$.

Equation $\Gamma^R(\tilde{z},\tilde{\tau})$ has the following properties:

1. $\lim\limits_{\tilde{z}\to 0}\Gamma^R(\tilde{z},\tilde{\tau}) = +\infty$, $\lim\limits_{\tilde{z}\to\frac{1-\tilde{\tau}}{v\tilde{\tau}}}\Gamma^R(\tilde{z},\tilde{\tau}) = (\frac{a}{\sigma}-1)(1-\tilde{\tau})^{a}\tilde{\tau}^{1-a}v^{1-a} - (\frac{1}{\sigma}-1)\delta$
2. $\lim\limits_{\tilde{\tau}\to 0}\Gamma^R(\tilde{z},\tilde{\tau}) = (\frac{a}{\sigma})(\tilde{z})^{a-1} - (\frac{1}{\sigma}-1)$, $\lim\limits_{\tilde{\tau}\to 1-a}\Gamma^R(\tilde{z},\tilde{\tau}) = (\frac{a}{\sigma})a(\tilde{z})^{a-1} - v(\tilde{z})^{a}(1-a) - (\frac{1}{\sigma}-1)\delta$
3. $\frac{\partial\Gamma^R(\tilde{z},\tilde{\tau})}{\partial\tilde{z}} = (\frac{a}{\sigma})(1-\tilde{\tau})(a-1)(\tilde{z})^{a-2} - va(\tilde{z})^{a-1}\tilde{\tau} < 0$
4. $\frac{\partial\Gamma^R(\tilde{z},\tilde{\tau})}{\partial\tilde{\tau}} = -(\frac{a}{\sigma})(\tilde{z})^{a-1} - v(\tilde{z})^{a} < 0$.

Equation $\Lambda^R(\tilde{z},\tilde{\tau})$ has the following properties:

1. $\lim\limits_{\tilde{z}\to 0}\Lambda^R(\tilde{z},\tilde{\tau}) = 0$, $\lim\limits_{\tilde{z}\to\frac{1-\tilde{\tau}}{\tilde{\tau}}}\Lambda^R(\tilde{z},\tilde{\tau}) = \frac{v}{\sigma}(\frac{(1-a)-\tilde{\tau}}{(\frac{1-\tilde{\tau}}{\tilde{\tau}})^{-a}}) > 0$ (since $\tilde{\tau} < 1-a$)
2. $\lim\limits_{\tilde{\tau}\to 0}\Lambda^R(\tilde{z},\tilde{\tau}) = \frac{v(1-a)}{\sigma\tilde{z}^{-a}} > 0$, $\lim\limits_{\tilde{\tau}\to 1-a}\Lambda^R(\tilde{z},\tilde{\tau}) = 0$
3. $\frac{\partial\Lambda^R(\tilde{z},\tilde{\tau})}{\partial\tilde{z}} = \frac{v((1-a)-\tilde{\tau})}{\sigma}(\frac{a\tilde{z}^{-a-1}}{\tilde{z}^{-2a}}) > 0$ (since $\tilde{\tau} < 1-a$)
4. $\frac{\partial\Lambda^R(\tilde{z},\tilde{\tau})}{\partial\tilde{\tau}} = -\frac{v\tilde{\tau}}{\sigma\tilde{z}^{-a}} < 0$.

From the properties of $\Gamma^R(\tilde{z},\tilde{\tau})$ and $\Lambda^R(\tilde{z},\tilde{\tau})$ it follows that they are monotone functions in both arguments for the domain of the variables.

Using all the information from equilibrium equations (29)-(31) and since both functions, $\Lambda^R(\tilde{z},\tilde{\tau})$, $\Gamma^R(\tilde{z},\tilde{\tau})$, are monotone in both arguments, if an intersection exists (see Proposition 1) it will be unique with only one bundle $(\tilde{z}, \tilde{\tau}) = (\tilde{z}^R, \tilde{\tau}^R)$ solving $V(\tilde{z}, \tilde{\tau})$ for any feasible tax rate. In turn, a unique solution for $\tilde{z}$ gives from (31) a unique solution for $\tilde{\omega}$.

To provide the necessary and sufficient conditions for existence we need to analyze the intersection of $\Lambda^R(\tilde{z}, \tilde{\tau})$ and $\Gamma^R(\tilde{z}, \tilde{\tau})$. In order for these functions to intersect in $\tilde{z}-$space we have the following: $\Lambda^R(\tilde{z}, \tilde{\tau})$ starts from 0 and ends at $\frac{v}{\sigma}(\frac{(1-a)-\tilde{\tau}}{(\frac{1-\tilde{\tau}}{\tilde{\tau}})^{-a}}) > 0$ and is increasing. $\Gamma^R(\tilde{z}, \tilde{\tau})$ starts from $+\infty$, ends at $(\frac{a}{\sigma} - 1)(1-\tilde{\tau})^a \tilde{\tau}^{1-a} v^{1-a} - (\frac{1}{\sigma} - 1)\delta$. Thus, for $\tilde{z} \to 0$ we have $\Gamma^R(\tilde{z}, \tilde{\tau}) > \Lambda^R(\tilde{z}, \tilde{\tau})$. As $\tilde{z} \to \frac{1-\tilde{\tau}}{v\tilde{\tau}}$ and $(\frac{a}{\sigma} - 1)(1-\tilde{\tau})^a \tilde{\tau}^{1-a} v^{1-a} - (\frac{1}{\sigma} - 1)\delta < \frac{v}{\sigma}(\frac{(1-a)-\tilde{\tau}}{(\frac{1-\tilde{\tau}}{v\tilde{\tau}})^{-a}})$ we get that $\Gamma^R(\tilde{z}, \tilde{\tau}) < \Lambda^R(\tilde{z}, \tilde{\tau})$. Since both functions are continuous then $(\frac{a}{\sigma} - 1)(1-\tilde{\tau})^a \tilde{\tau}^{1-a} v^{1-a} - (\frac{1}{\sigma} - 1)\delta < \frac{1}{\sigma}(\frac{(1-a)-\tilde{\tau}}{(\frac{1-\tilde{\tau}}{v\tilde{\tau}})^{-a}})$ is a sufficient condition for the existence of a solution in $\tilde{z}$-space, which holds for any $\tilde{\tau} < 1 - a$ if $\sigma < a$.

A similar approach holds for the $\tilde{\tau}-$space. In order for the above functions to intersect in $\tilde{\tau}-$space we have the following: $\Lambda^R(\tilde{z}, \tilde{\tau})$ starts from $\frac{v(1-a)}{\sigma\tilde{z}^{-a}}$, ends at zero and is decreasing. $\Gamma^R(\tilde{z}, \tilde{\tau})$ starts from $(\frac{a}{\sigma})(\tilde{z})^{a-1} - (\frac{1}{\sigma} - 1)$, ends at $(\frac{a}{\sigma})a(\tilde{z})^{a-1} - v(\tilde{z})^a(1-a) - (\frac{1}{\sigma} - 1)\delta$ and is increasing. Thus, for $\tilde{\tau} \to 0$ we have $\Lambda^R(\tilde{z}, \tilde{\tau}) = \frac{v(1-a)}{\sigma\tilde{z}^{-a}} > \Gamma^R(\tilde{z}, \tilde{\tau}) = 0$. As $\tilde{\tau} \to 1 - a$ we have $\Lambda^R(\tilde{z}, \tilde{\tau}) = 0 < \Gamma^R(\tilde{z}, \tilde{\tau}) = \frac{v}{\sigma}(\frac{(1-a)-\tilde{\tau}}{(\frac{1-\tilde{\tau}}{\tilde{\tau}})^{-a}})$.$\blacksquare$

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Table 1. Changes in σ and CDE properties

σ	g_1^{CDE}	g_2^{CDE}	$\bar{z}_1$	$\bar{z}_2$	$\bar{\omega}_1$	$\bar{\omega}_2$	$\bar{\rho}_1$	$\bar{\rho}_2$
Multiplicity								
0.1	0.0137	0.0713	4.63	28.63	0.216	0.006	0.1014	0.0192
0.15	0.0136	0.0720	4.60	29.09	0.217	0.005	0.1011	0.0151
0.2	0.0134	0.0728	4.57	29.52	0.218	0.004	0.1000	0.0110
0.25	0.0133	0.0735	4.54	29.96	0.220	0.002	0.1007	0.0068
0.3	0.0132	0.0742	4.51	30.39	0.221	0.001	0.1005	0.0026
Uniqueness								
0.4	0.0129	-	4.45	-	0.2226	-	0.1001	-
0.5	0.0127	-	4.40	-	0.2244	-	0.0997	-
0.6	0.0123	-	4.30	-	0.2279	-	0.0989	-
0.7	0.0121	-	4.25	-	0.2295	-	0.0986	-
0.8	0.0119	-	4.21	-	0.2311	-	0.0982	-

Notes: $a = 0.5$, $\delta = 0.025$, $\tau = 0.45$, $b = 0.1$, $v = 0.04$, $\check{\rho} = 0.001$.

Table 2. Ramsey equilibrium selection under CDE multiplicity

σ	g^R	$\tilde{\tau}$	$\tilde{z}$	$\tilde{\omega}$	$\tilde{\rho}$
0.1	0.071	0.419	32.84	0.005	0.018
0.15	0.072	0.433	31.24	0.004	0.014
0.2	0.073	0.449	29.61	0.003	0.011
0.25	0.074	0.467	27.93	0.002	0.007
0.3	0.075	0.486	26.20	0.001	0.002

Notes: $a = 0.5$, $\delta = 0.025$, $b = 0.1$, $v = 0.04$, $\check{\rho} = 0.001$.

Table 3. Changes in σ and GM allocation

σ	g^{GM}	$\hat{\tau}$	$\hat{z}$	$\hat{\omega}$	$\hat{\rho}$
0.1	0.071	0.455	27.88	0.0069	0.020
0.2	0.073	0.473	26.67	0.004	0.012
0.3	0.074	0.494	25.39	0.063	0.003
0.4	0.033	0.773	3.486	0.042	0.023
0.5	0.030	0.763	3.283	0.102	0.025
0.6	0.028	0.754	3.134	0.229	0.027
0.7	0.027	0.747	3.015	0.313	0.029
0.8	0.026	0.741	2.917	0.381	0.030
0.9	0.025	0.736	2.833	0.439	0.031
1	0.024	0.731	2.760	0.491	0.032

Notes: $a = 0.5$, $\delta = 0.025$, $b = 0.4$, $v = 0.04$, $\check{\rho} = 0.001$.

Table 4. Changes in b and GM allocation

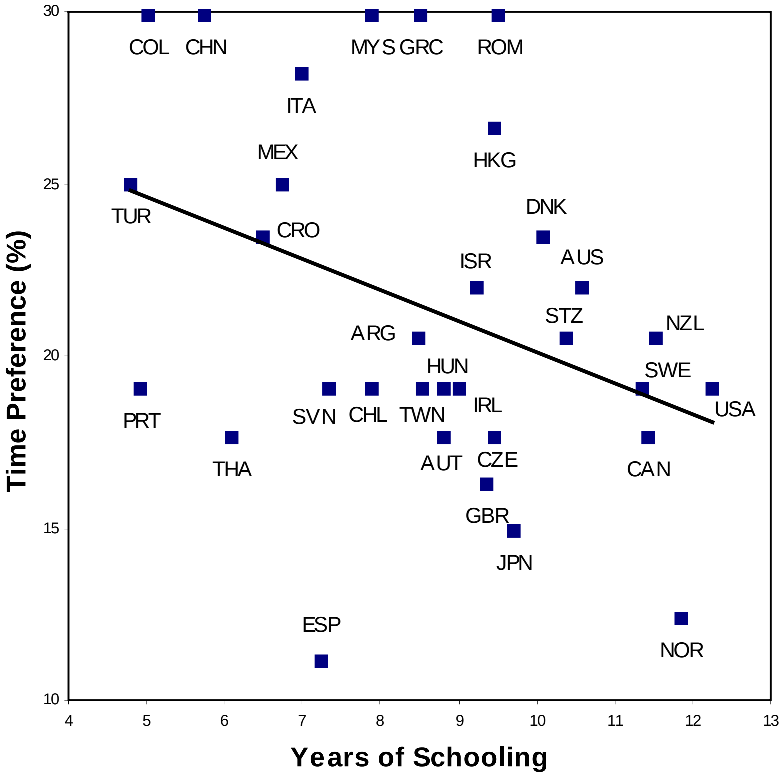
b	g^{GM}	$\hat{\tau}$	$\hat{z}$	$\hat{\omega}$	$\frac{\hat{C}}{H}$	$\hat{\rho}$
Multiplicity in CDE ($\sigma = 0.2$)						
0.1	0.073	0.473	26.67	0.004	0.107	0.012
0.4	0.074	0.494	25.28	0.001	0.025	0.011
0.7	0.075	0.497	25.15	0.0005	0.012	0.010
Uniqueness in CDE ($\sigma = 1$)						
0.1	0.023	0.731	2.76	0.113	0.311	0.032
0.4	0.007	0.844	0.92	0.130	0.120	0.048
0.7	0.001	0.878	0.56	0.135	0.076	0.054

Notes: $a = 0.5$, $\delta = 0.025$, $v = 0.04$, $\check{\rho} = 0.001$.

Table 5. Changes in σ and b and summarized GM allocation responses

	g^{GM}	$\hat{\tau}$	$\hat{z}$	$\hat{\omega}$	$\frac{\hat{C}}{H}$	$\hat{\rho}$
increase in σ for low σ	(+)	(+)	(−)	(+)	(−)	(−)
increase in σ for high σ	(−)	(−)	(−)	(+)	(+)	(+)
increase in b for $\sigma = 0.2$	(+)	(+)	(−)	(−)	(−)	(−)
increase in b for $\sigma = 1$	(−)	(+)	(−)	(+)	(−)	(+)

Figure 1. Time Preference and Human Capital



Source: Barro and Lee (2001), Wang et al. (2010).