Companion Appendix to
"Green Spending Reforms, Growth and Welfare with Endogenous Subjective Discounting"
(Not for Publication)

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1 Transitional dynamics and stability analysis

Linearizing (11a)-(11c) around (12a)-(12c) implies that the local dynamics are approximated by the linear system:

$$\begin{bmatrix} \dot{\omega} \\ \dot{z} \\ \dot{x} \end{bmatrix} = \begin{bmatrix} J_{\omega\omega} & J_{z\omega} & J_{x\omega} \\ J_{\omega z} & J_{zz} & J_{xz} \\ J_{\omega x} & J_{zx} & J_{xx} \end{bmatrix} \begin{bmatrix} \omega - \hat{\omega} \\ z - \hat{z} \\ x - \hat{x} \end{bmatrix}$$

where the elements of the Jacobian matrix, J, evaluated at the long run are:

$J_{\omega\omega} \equiv \frac{\vartheta\dot{\omega}}{\vartheta\omega} = \hat{\omega}\left[1 - \frac{\rho'(.)\hat{z}\hat{x}}{1-\nu(1-\sigma)}\right] \gtreqless 0,$

$J_{z\omega} \equiv \frac{\vartheta\dot{\omega}}{\vartheta z} = \frac{\hat{\omega}}{[1-\nu(1-\sigma)]\hat{z}}[a\,(1-\nu)\,(1-\sigma)\,\Theta(\tau,b)\hat{z}^a\hat{x} + (1-a)(1-\nu\,(1-\sigma)-a)(1-\tau)\hat{z}^{a-1} - \rho'(.)\hat{\omega}\hat{z}\hat{x}] \gtreqless 0,$

$J_{x\omega} \equiv \frac{\vartheta\dot{\omega}}{\vartheta x} = \frac{\hat{\omega}}{1-\nu(1-\sigma)}[(1-\nu)\,(1-\sigma)\,\Theta(\tau,b)\hat{z}^a - \rho'(.)\hat{\omega}\hat{z}] < 0,$

$J_{\omega z} \equiv \frac{\vartheta\dot{z}}{\vartheta\omega} = -\hat{z} < 0,$

$J_{zz} \equiv \frac{\vartheta\dot{z}}{\vartheta z} = -(1-a)(1-\tau)\hat{z}^{a-1} - ab\tau\hat{z}^a < 0,$

$J_{xz} \equiv \frac{\vartheta\dot{z}}{\vartheta x} = 0,$

$J_{\omega x} \equiv \frac{\vartheta\dot{x}}{\vartheta\omega} = 0,$

$J_{zx} \equiv \frac{\vartheta\dot{x}}{\vartheta z} = a\frac{\hat{x}}{\hat{z}}(\delta + \delta_N) > 0,$

$J_{xx} \equiv \frac{\vartheta\dot{x}}{\vartheta x} = -\Theta(\tau,b)\hat{z}^a\hat{x} > 0.$

The trace and the determinant of J, $trace(J) = J_{\omega\omega} + J_{zz} + J_{xx}$ and $det(J) = J_{\omega\omega}J_{zz}J_{xx} - J_{z\omega}J_{\omega z}J_{xx} + J_{x\omega}J_{\omega z}J_{zx}$ have ambiguous signs. Due to the complexity for the computation of these signs, we provide numerical results for the eigenvalues of J, denoted by ε. First, as a benchmark we perform the computations in the case of exogenous RTP ($\rho'(.) = 0$). In this case, the long-run equilibrium is unique and the dynamic system has two positive and one negative eigenvalues (results available upon request). Hence it follows that there exist locally a one-dimensional stable and a two-dimensional unstable manifolds, since we have one jump variable (ω) and two state/predetermined variables (z, x).

Next, we perform the computations in the case of endogenous RTP ($\rho'(.) > 0$) with regard to the long-run equilibria displayed in Table 2. The findings, reported in Table A1, are similar (i.e. one negative and two positive eigenvalues) for the 'bad' equilibrium, which corresponds to the unique equilibrium of the exogenous RTP case. However, in the 'good' equilibrium there are two negative and one positive eigenvalues, which implies that this regime is saddle-path stable.

2 Equations (19a)-(19d) in the Ramsey allocation

The Hamiltonian of the problem is given by:

$$H^R = \frac{\left(C^{\nu}N^{1-\nu}\right)^{1-\sigma}}{1-\sigma}e^{-\Delta} + \tilde{\lambda}_1\left[(1-\tau)K^aK_g^{1-a} - C - \delta K\right] + \tilde{\lambda}_2[G - \delta K_g]$$
$$+ \tilde{\lambda}_3\left[\delta_N N - sK^aK_g^{1-a} + \theta E\right] + \tilde{\lambda}_4\left[\tau K^aK_g^{1-a} - G - E\right] + \tilde{\lambda}_5\rho\left(\frac{C}{N}\right)$$

The optimality conditions, as given by equations (18a)-(18h), and the competitive-equilibrium growth rates, given by (9a)-(9d), completely characterize the solution of the Ramsey problem.

2.1 Derivation of (19a)

From (18a):

$$\nu C^{\nu(1-\sigma)-1}N^{(1-\nu)(1-\sigma)} - \lambda_1 + \frac{1}{N}\lambda_5\rho'(\frac{C}{N}) = 0$$

$$\overset{(18d)}{\Rightarrow} \frac{C}{N}\lambda_5\rho'(\frac{C}{N}) = \lambda_4 K_g\frac{K}{K_g}\frac{C}{K} - \nu C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}$$

$$\Rightarrow \frac{C}{N}\lambda_5\rho'(\frac{C}{N}) = \chi\omega z - \nu C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} \tag{A1}$$

Also, from (18c):

$$\Rightarrow (1-\nu)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)-1} + \lambda_3\delta_N - \lambda_5\frac{C}{N^2}\rho'(\frac{C}{N}) = -\dot{\lambda}_3 + \lambda_3\rho(\frac{C}{N})$$

$$\Rightarrow \dot{\lambda}_3 N = -(1-\nu)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} - \lambda_3 N\delta_N + \lambda_5\frac{C}{N}\rho'(\frac{C}{N}) + \lambda_3 N\rho(\frac{C}{N})$$

$$\overset{(A1)}{\Rightarrow} \dot{\lambda}_3 N = -(1-\nu)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} - \lambda_3 N\delta_N + \chi\omega z - \nu C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)} + \lambda_3 N\rho(\frac{C}{N})$$

$$\Rightarrow \dot{\lambda}_3 N = -C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} - \phi\delta_N + \chi\omega z + \phi\rho(\frac{C}{N}) \tag{A2}$$

From (18d)-(18f) we have $\lambda_1 = \lambda_2 = \lambda_4$, $\dot{\lambda}_1 = \dot{\lambda}_2 = \dot{\lambda}_4 = \theta\dot{\lambda}_3$. Then (18b) implies:

$$\dot{\lambda}_2 = -\lambda_1(1-a)(1-\tau)K^a K_g^{-a} + \lambda_2\delta + \lambda_2(1-a)sK^a K_g^{-a} - \lambda_4(1-a)\tau K^a K_g^{-a} + \lambda_2\rho(\frac{C}{N})$$

and (18c) implies:

$$\dot{\lambda}_3 = -(1-\nu)C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)-1} - \lambda_3\delta_N + \lambda_5\frac{C}{N^2}\rho'(\frac{C}{N}) + \lambda_3\rho$$

$$\Rightarrow \theta\dot{\lambda}_3 = -\theta(1-\nu)C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)-1} - \theta\lambda_3\delta_N + \theta\lambda_5\frac{C}{N^2}\rho'(\frac{C}{N}) + \theta\lambda_3\rho(\frac{C}{N})$$

Then, equating $\theta\dot{\lambda}_3 = \dot{\lambda}_2$ we get:

$$\Rightarrow -\theta(1-\nu)C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} - \theta\lambda_3\delta_N N + \theta\lambda_5\frac{C}{N}\rho'(\frac{C}{N}) + \theta\lambda_3 N\rho(\frac{C}{N}) = -\lambda_1 N(1-a)(1-\tau)K^a K_g^{-a}$$
$$+ \lambda_2 N(1-a)sK^a K_g^{-a} - \lambda_4 N(1-a)\tau K^a K_g^{-a} + \lambda_2 N\delta + \lambda_2 N\rho(\frac{C}{N})$$

$$\overset{\lambda_1=\lambda_2=\lambda_4,\ \dot{\lambda}_1=\dot{\lambda}_2=\dot{\lambda}_4=\theta\dot{\lambda}_3}{\Rightarrow} -\theta(1-\nu)C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} - \lambda_4 N\delta_N + \theta\lambda_5\frac{C}{N}\rho'(\frac{C}{N}) = -\lambda_4 N(1-a)K^a K_g^{-a}$$
$$+ \lambda_4 N(1-a)sK^a K_g^{-a} + \lambda_4 N\delta$$

$$\overset{\chi N \equiv \lambda_4 K_g N \Rightarrow \lambda_4 N = \chi\frac{N}{K_g} \Rightarrow \lambda_4 N = \frac{\chi}{x}}{\Rightarrow} -\theta(1-\nu)C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} + \frac{\chi}{x}\delta_N + \theta\lambda_5\frac{C}{N}\rho'(\frac{C}{N}) = -\frac{\chi}{x}(1-a)z^a(1-s) + \frac{\chi}{x}\delta$$

$$\overset{(A1)}{\Rightarrow} -\theta(1-\nu)C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} + \frac{\chi}{x}\delta_N + \theta\chi\omega z - \theta\nu C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} = -\frac{\chi}{x}(1-a)z^a(1-s) + \frac{\chi}{x}\delta$$

$$\Rightarrow C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} = \frac{\chi}{\theta x}[\delta_N - \delta + (1-a)z^a(1-s)] + \chi\omega z \tag{A3}$$

Then substituting (A2) and (9d) in $\dot{\phi} \equiv \dot{\lambda}_3 N + \lambda_3\dot{N}$ we get that:

$$\Rightarrow \dot{\phi} = -C^{\nu(1-\sigma)} N^{(1-\nu)(1-\sigma)} + \chi\omega z + \rho(\omega z x)\phi + \Theta(\tau, b)z^a x\phi$$

$$\overset{(A3)}{\Rightarrow} \frac{\dot{\phi}}{\phi} = \frac{\chi}{\theta x \phi}\left[-\delta_N + \delta - (1-a)z^a(1-s)\right] + \rho(\omega z x) + \Theta(\tau, b) z^a x \tag{19a}$$

2.2 Derivation of (19b)

Using $\lambda_1 = \lambda_2 = \lambda_4$ and $\dot{\lambda}_1 = \dot{\lambda}_2 = \dot{\lambda}_4 = \dot{\lambda}_3\theta$ in (18b) we get:

$$\dot{\lambda}_4 = -\lambda_4(1-a)(1-\tau)K^a K_g^{-a} + \lambda_4\delta + \frac{1}{\theta}\lambda_4(1-a)sK^a K_g^{-a} - \lambda_4(1-a)\tau K^a K_g^{-a} + \lambda_4\rho(\frac{C}{N})$$

$$\Rightarrow \dot{\lambda}_4 = -\lambda_4(1-a)K^a K_g^{-a} + \lambda_4\delta + \frac{1}{\theta}\lambda_4(1-a)sK^a K_g^{-a} + \lambda_4\rho(\frac{C}{N}) \tag{A4}$$

Then from $\dot{\chi} = \dot{\lambda}_4 K_g + \lambda_4 \dot{K}_g$ it follows that:

$$\overset{(A4),\,(9c)}{\Rightarrow} \frac{\dot{\chi}}{\chi} = -\left(1 - \frac{s}{\theta}\right)(1-a)z^a + b\tau z^a + \rho(\omega z x) \tag{19b}$$

2.3 Derivation of (19c)

From (18g) we have in the long run:

$$\frac{\dot{\lambda}_5}{\lambda_5} = \frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{(1-\sigma)\lambda_5} + \rho(\frac{C}{N}) = 0 \Rightarrow \lambda_5 = -\frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{(1-\sigma)\rho(\frac{C}{N})} \tag{A5}$$

Substituting (A5) in (A1) we get that:

$$-\frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{(1-\sigma)\rho(\omega z x)}\frac{C}{N}\rho'(\frac{C}{N}) = \chi\omega z - \nu C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}$$

$$\Longrightarrow -C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}x\omega z\rho'(x\omega z) = (1-\sigma)\rho(\omega z x)\chi\omega z - \nu(1-\sigma)\rho(\omega z x)C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}$$

$$\Longrightarrow C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}\left[\nu(1-\sigma)\rho(\omega z x) - x\omega z\rho'(x\omega z)\right] - (1-\sigma)\rho(\omega z x)\chi\omega z = 0$$

$$\overset{(A3)}{\Longrightarrow} \left[\frac{1}{\theta x}(\delta_N - \delta) + \omega z + \frac{1}{\theta x}(1-a)(1-s)z^a\right]\left[\nu(1-\sigma)\rho(\omega z x) - x\omega z\rho'(x\omega z)\right] - (1-\sigma)\rho(\omega z x)\omega z = 0 \tag{19c}$$

2.4 Derivation of (19d)

From (18h) we have:

$$\frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{1-\sigma} + \lambda_1\dot{K} + \lambda_2\dot{K}_g + \lambda_3\dot{N} + \lambda_5\rho(\cdot) = 0$$

$$\overset{(18d),\,(18e)}{\Rightarrow} \frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{1-\sigma} + \lambda_4 K_g\frac{K}{K_g}\frac{\dot{K}}{K} + \lambda_4 K_g\frac{\dot{K}_g}{K_g} + \lambda_3 N\frac{\dot{N}}{N} + \lambda_5\rho(\cdot) = 0$$

$$\overset{(A5)}{\Rightarrow} \frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{1-\sigma} + \chi z\frac{\dot{K}}{K} + \chi\frac{\dot{K}_g}{K_g} + \phi\frac{\dot{N}}{N} - \frac{C^{\nu(1-\sigma)}N^{(1-\nu)(1-\sigma)}}{(1-\sigma)} = 0$$

$$\Rightarrow (\chi z + \chi + \phi)\frac{\dot{K}_g}{K_g} = 0 \overset{(9c)}{\Rightarrow} (\chi z + \chi + \phi)(b\tau z^a - \delta) = 0 \tag{19d}$$

Table A1. Eigenvalues of the Jacobian matrix

	'Bad' Equilibrium			'Good' Equilibrium		
σ	ε_1	ε_2	ε_3	ε_1	ε_2	ε_3
0.1	-24.7649	1.7677	0.0158	-1.2587	0.3327	-0.2556
0.2	-24.8602	1.6197	0.0161	-0.9518	0.3134	-0.2875
0.3	-24.9850	1.4922	0.0163	-0.5946	-0.3683	0.2916
0.4	-25.1329	1.3812	0.0166		-	
0.5	-25.2992	1.2837	0.0169		-	
0.6	-25.4802	1.1973	0.0171		-	
0.7	-25.6733	1.1204	0.0174		-	
0.8	-25.8761	1.0513	0.0176		-	
0.9	-26.0869	0.9890	0.0179		-	
1.0	-26.3044	0.9325	0.0182		-	

Note: $a = 0.5$, $\delta = 0.025$, $\delta_N = 0.5$, $\theta = 1$, $s = 0.5$, $\gamma = 1$, $\check{\rho} = 0.04$, $\tau = 0.6$, $b = 0.5$, $\nu = 0.5$.