THE ADAPTIVE PERFECTLY MATCHED LAYER METHOD FOR TIME-HARMONIC ACOUSTIC AND ELECTROMAGNETIC SCATTERING PROBLEMS

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Abstract. The recently introduced perfectly matched layer (PML) method is an efficient method to reduce the exterior wave propagation problem which is defined in the unbounded domain to the problem in the bounded domain. Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that would attenuate all the waves that propagate from inside the computational domain. We report our recent efforts in developing the adaptive PML method for solving the time-harmonic acoustic and electromagnetic wave scattering problems. The method uses the a posteriori error estimate to determine the PML parameters such as the thickness of the layer and the artificial medium property. Combined with the adaptive finite element method, the adaptive PML method provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer. We will consider the adaptive uniaxial PML method for the Helmholtz scattering problems in stratified medium and the electromagnetic scattering problems.

1. Introduction. In this paper we report our recent efforts in developing the adaptive perfectly matched layer (PML) method for solving the acoustic and electromagnetic wave scattering problems. One of the fundamental problems in the efficient simulation of the wave propagation is the reduction of the exterior problem which is defined in the unbounded domain to the problem in the bounded domain. One popular method is to approximate the exact non-local Dirichlet-to-Neumann boundary condition by the absorbing boundary conditions which have been proposed and studied in the literature, see the review papers Givoli [21], Tsynkov [29], Hagstrom [22] and the references therein. An interesting alternative to the method of absorbing boundary conditions is the PML method which now attracts ever increasing interests in the engineering applications.

Since the work of Bérenger [5] which proposed a PML technique for solving the time-dependent Maxwell equations in the Cartesian coordinates, various constructions of PML absorbing layers have been proposed and studied in the literature (cf. e.g. Turkel and Yefet [30], Teixeira and Chew [28] for the reviews). Under the assumption that the exterior solution is composed of outgoing waves only, the basic idea of the PML technique is to surround the computational domain by a layer of finite thickness with specially designed model medium that would attenuate all the waves that propagate from inside the computational domain.

Although the tremendous attention and success in the application of PML methods in the engineering literature, there are few mathematical results on the convergence of the PML methods. The convergence of the PML method is studied in Lassas and Somersalo [24], Hohage et al [23] for the acoustic scattering problems.

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and in Bao and Wu [4], Bramble and Pasciak [6] for the electromagnetic scattering problems. It is proved in [24, 23, 6] that the PML solution converges exponentially to the solution of the original scattering problem as the thickness of the PML layer tends to infinity. We remark that in practical applications involving PML techniques, one cannot afford to use a very thick PML layer if uniform meshes are used because it requires excessive grid points and hence more computer time and more storage. On the other hand, a thin PML layer requires a rapid variation of the artificial material property which deteriorates the accuracy if too coarse mesh is used in the PML layer.

The adaptive PML method was first proposed in Chen and Wu [12] for a scattering problem by periodic structures (the grating problem) in which one uses the a posteriori error estimate to determine the PML parameters. It was extended in Chen and Liu [11] to the acoustic scattering problem and in Chen and Chen [9] to the time-harmonic Maxwell scattering problems. The idea of adaptive PML method is extended to the uniaxial PML method in Chen and Wu [13] for the acoustic scattering problems which uses rectangular domain to define the PML problem and thus provides greater flexibility and efficiency in dealing with problems involving anisotropic scatterers.

A posteriori error estimates are computable quantities in terms of the discrete solution and data that measure the actual discrete errors without the knowledge of exact solutions. The adaptive finite element method based on a posteriori error estimates initiated in Babuska and Rheinboldt [5] provides a systematic way to achieve the optimal computational complexity by refining the mesh according to the local a posteriori error estimator on the elements. Combined with the adaptive finite element method, the adaptive PML technique provides a complete numerical strategy to solve the scattering problems in the framework of finite element which produces automatically a coarse mesh size away from the fixed domain and thus makes the total computational costs insensitive to the thickness of the PML absorbing layer.

In this paper we will first consider the general framework to derive a posteriori error estimate which is crucial in developing the adaptive PML method through the example of solving the Helmholtz scattering problem with circular PML layers. We show that the crucial role played in the analysis is the proof of the exponential decay of the PML extension and the stability of the Dirichlet PML problem in the PML layer. In Section 3 we consider the uniaxial PML method for the Helmholtz equations in the layered medium. In Section 4 we consider the adaptive uniaxial PML method for electromagnetic scattering problems.

2. The adaptive PML method. In this section we describe the a posteriori error analysis and the adaptive PML method for solving Helmholtz-type scattering problems with perfectly conducting boundary:

\begin{align*}
\Delta u + k^2 u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \overline{D}, \\
\frac{\partial u}{\partial n_D} &= -g \quad \text{on } \Gamma_D, \\
\sqrt{r} \left( \frac{\partial u}{\partial r} - ik u \right) &\to 0 \quad \text{as } r = |x| \to \infty.
\end{align*}
Here \( D \subset \mathbb{R}^2 \) is a bounded domain with Lipschitz boundary \( \Gamma_D \), \( g \in H^{-1/2}(\Gamma_D) \) is determined by the incoming wave, and \( \mathbf{n}_D \) is the unit outer normal to \( \Gamma_D \). is the Sommerfeld radiation condition. We assume the wave number \( k \in \mathbb{R} \) is a constant. We remark that the results in this section can be easily extended to solve the scattering problems with other boundary conditions such as Neumann or the impedance boundary condition on \( \Gamma_D \), or to solve the acoustic wave propagation through inhomogeneous media with a variable wave number \( k(x) \) inside some bounded domain.

Let \( D \) be contained in the interior of the circle \( B_R = \{ x \in \mathbb{R}^2 : |x| < R \} \). We start by introducing an equivalent variational formulation of (2.1)-(2.3) in the bounded domain \( \Omega_R = B_R \setminus D \). In the domain \( \mathbb{R}^2 \setminus B_R \), the solution \( u \) of \( \mathbf{2.1}-\mathbf{2.3} \) can be written under the polar coordinates as follows:

\[
\begin{align*}
(2.4) \quad u(r, \theta) &= \sum_{n \in \mathbb{Z}} \frac{H_n^{(1)}(kr)}{H_0^{(1)}(kR)} \hat{u}_n e^{in\theta}, \quad \hat{u}_n = \frac{1}{2\pi} \int_0^{2\pi} u(R, \theta)e^{-in\theta} d\theta.
\end{align*}
\]

where \( H_n^{(1)} \) is the Hankel function of the first kind and order \( n \). The series in (2.4) converges uniformly for \( r > R \) (cf. e.g. Colton and Kress [19]). Let \( \mathcal{T} : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R) \), where \( \Gamma_R = \partial B_R \), be the Dirichlet-to-Neumann operator defined as follows: for any \( f \in H^{1/2}(\Gamma_R) \),

\[
(2.5) \quad \mathcal{T}f = \sum_{n \in \mathbb{Z}} k \frac{H_n^{(1)}(kr)}{H_0^{(1)}(kR)} \hat{f}_n e^{in\theta}, \quad \hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f e^{-in\theta} d\theta.
\]

It is known that \( \mathcal{T} \) is well-defined and the solution \( u \) written as in (2.4) satisfies \( \frac{\partial u}{\partial r} \big|_{\Gamma_R} = \mathcal{T}u \).

Let \( a : H^1(\Omega_R) \times H^1(\Omega_R) \to \mathbb{C} \) be the sesquilinear form:

\[
(2.6) \quad a(\varphi, \psi) = \int_{\Omega_R} \left( \nabla \varphi \cdot \nabla \overline{\psi} - k^2 \varphi \overline{\psi} \right) dx - \langle \mathcal{T} \varphi, \psi \rangle_{\Gamma_R},
\]

where \( \langle \cdot, \cdot \rangle_{\Gamma_R} \) stands for the inner product on \( L^2(\Gamma_R) \) or the duality pairing between \( H^{-1/2}(\Gamma_R) \) and \( H^{1/2}(\Gamma_R) \). Similar notation applies for \( \langle \cdot, \cdot \rangle_{\Gamma_D}, \langle \cdot, \cdot \rangle_{\Gamma_R} \). The scattering problem (2.1)-(2.3) is equivalent to the following weak formulation (cf. e.g. [19]): Given \( g \in H^{-1/2}(\Gamma_D) \), find \( u \in H^1(\Omega_R) \) such that

\[
(2.7) \quad a(u, \psi) = \langle g, \psi \rangle_{\Gamma_D}, \quad \forall \psi \in H^1(\Omega_R).
\]

The existence of a unique solution of the variational problem (2.7) is known (cf. e.g. [19], McLean [25]). Then the general theory in Babuška and Aziz [2, Chapter 5] implies that there exists a constant \( \mu > 0 \) such that the following inf-sup condition holds:

\[
(2.8) \quad \sup_{0 \neq \varphi \in H^1(\Omega_R)} \frac{|a(\varphi, \psi)|}{\| \psi \|_{H^1(\Omega_R)}} \geq \mu \| \varphi \|_{H^1(\Omega_R)}, \quad \forall \varphi \in H^1(\Omega_R).
\]

Now we turn to the introduction of the absorbing PML layer. We surround the domain \( \Omega_R \) with a layer \( \Omega_{\text{PML}} = \{ x \in \mathbb{R}^2 : R < |x| < \rho \} \). Let \( \alpha(r) = 1 + i \sigma(r) \) be the model medium property which satisfies

\[
\sigma \in L^\infty(\mathbb{R}), \quad \sigma \geq 0, \quad \text{and} \quad \sigma = 0 \quad \text{for} \; r \leq R.
\]
In practical applications, the medium property is usually taken as the power function

\[ \sigma = \sigma_0 \left( \frac{r - R}{\rho - R} \right)^m \]  

for some constant \( \sigma_0 > 0 \) and some integer \( m \geq 0 \). Denote by \( \tilde{r} \) the complex radius defined by

\[ \tilde{r} = \tilde{r}(r) = \begin{cases} r & \text{if } r \leq R, \\ \int_0^r \alpha(t) dt = r \beta(r) & \text{if } r \geq R. \end{cases} \]  

We follow the method of coordinate stretching Chew and Weedon [17], Collion and Monk [18] to introduce the PML equation. Since \( H_n^{(1)}(z) \sim \sqrt{\frac{2}{\pi z}} e^{i(z - \pi/2 - \pi/4)} \) as \( |z| \to \infty \), for any function \( f \in H^{1/2}(\Gamma_R) \), we consider the PML extension \( E(f)(x) \) given by

\[ E(f)(x) = \sum_{n \in \mathbb{Z}} \frac{H_n^{(1)}(k \tilde{r})}{H_n^{(1)}(kR)} \hat{f}_n e^{i n \theta}, \quad \hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f e^{-i n \theta} d\theta. \]  

It is easy to see that \( E(f)|_{\Gamma_R} = f \) on \( \Gamma_R \). For the solution \( u \) of the scattering problem \( (2.7) \), let \( w = E(u|_{\Gamma_R}) \) be the PML extension of \( u|_{\Gamma_R} \). It is obvious that \( w \) satisfies the equation

\[ \nabla \cdot (A \nabla w) + \alpha \beta k^2 w = 0 \quad \text{in } \Omega^{\text{PML}}, \]

where \( A = A(x) \) is a matrix which satisfies, in polar coordinates,

\[ \nabla \cdot (A \nabla) = \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{\beta r}{\alpha} \frac{\partial}{\partial r} \right) + \frac{\alpha}{\beta r^2} \frac{\partial^2}{\partial \theta^2}. \]
The PML solution \( \hat{u} \) in \( \Omega_\rho = B_\rho \setminus \bar{D} \) is defined as the solution of the following system

\[
(2.12a) \quad \nabla \cdot (A \nabla \hat{u}) + \alpha \beta k^2 \hat{u} = 0 \quad \text{in} \; \Omega_\rho,
\]

\[
(2.12b) \quad \frac{\partial \hat{u}}{\partial n_D} = -g \quad \text{on} \; \Gamma_D, \quad \hat{u} = 0 \quad \text{on} \; \Gamma_\rho.
\]

### 2.1. The convergence of the PML method.

Now we describe the key ingredients in studying the convergence of the PML method. Let \( e = u - \hat{u} \) be the error between the solution of the original scattering problem and the solution of the PML problem. It is clear that \( e \) satisfies

\[
\Delta e + k^2 e = 0 \quad \text{in} \; \Omega_R = B_R \setminus \bar{D},
\]

\[
\frac{\partial e}{\partial n_D} = 0 \quad \text{on} \; \Gamma_D, \quad \frac{\partial e}{\partial r} = \mathbb{T} u - \hat{\mathbb{T}} \hat{u} \quad \text{on} \; \Gamma_R.
\]

Here \( \mathbb{T} \) is the Dirichlet-to-Neumann operator in (2.5) and \( \hat{\mathbb{T}} : H^{1/2}(\Gamma_R) \to H^{-1/2}(\Gamma_R) \) is the approximate Dirichlet-to-Neumann operator which we now define. Given \( f \in H^{1/2}(\Gamma_R) \), \( \hat{\mathbb{T}} f = \left. \frac{\partial \zeta}{\partial r} \right|_{\Gamma_R} \), where \( \zeta \in H^1(\Omega_{\text{PML}}) \) satisfies

\[
(2.13) \quad \nabla \cdot (A \nabla \zeta) + \alpha \beta k^2 \zeta = 0 \quad \text{in} \; \Omega_{\text{PML}} = B_\rho \setminus \bar{B}_R,
\]

\[
(2.14) \quad \zeta = f \quad \text{on} \; \Gamma_R, \quad \zeta = 0 \quad \text{on} \; \Gamma_\rho.
\]

It is obvious that the estimation of the error \( e \) reduces to the estimate of \( \mathbb{T} f - \hat{\mathbb{T}} f \) for any given \( f \in H^{1/2}(\Gamma_R) \).

Introduce the sesquilinear form \( c : H^1(\Omega_{\text{PML}}) \times H^1(\Omega_{\text{PML}}) \to \mathbb{C} \) as

\[
c(\varphi, \psi) = \int_R^\rho \int_0^{2\pi} \left( \frac{\beta r}{\alpha} \frac{\partial \varphi}{\partial r} \frac{\partial \bar{\psi}}{\partial r} + \frac{\alpha}{\beta r} \frac{\partial \varphi}{\partial \theta} \frac{\partial \bar{\psi}}{\partial \theta} - \alpha \beta k^2 r \varphi \bar{\psi} \right) dr d\theta
\]

Then the weak formulation of (2.13) – (2.14) is: Find \( \zeta \in H^1(\Omega_{\text{PML}}) \) such that \( \zeta = f \) on \( \Gamma_R \), \( \zeta = 0 \) on \( \Gamma_\rho \), and

\[
(2.15) \quad c(\zeta, v) = 0, \quad \forall \, v \in H^1_0(\Omega_{\text{PML}}).
\]

The PML problem (2.13)–(2.14) in the layer is a compact perturbation of a coercive elliptic operator in \( H^1_0(\Omega_{\text{PML}}) \). It follows from the analytic Fredholm alternative theorem that it exists a unique solution for every real \( k \) except possibly for a discrete set of values of \( k \). It is proved in [11] that the PML problem in the layer has a unique solution provided \( \sigma_0 \) is sufficiently large. The following lemma which ensures the stability of the PML problem in the layer for the constant medium property is motivated by the analysis in Chen [10]. We remark our extensive numerical experiences [14] indicate that PML problems with constant medium property have advantages in the preconditioning of the discrete PML matrices.

**Lemma 2.1.** Let \( \sigma(r) = \sigma_0 \) for \( r \geq R \), where \( \sigma_0 \) is a positive constant. Then the problem (2.15) has a unique weak solution and it holds that, for any \( \varphi \in H^1_0(\Omega_{\text{PML}}) \),

\[
(2.16) \quad \sup_{0 \neq \psi \in H^1_0(\Omega_{\text{PML}})} \frac{|c(\varphi, \psi)|}{\|\psi\|_{*,\Omega_{\text{PML}}} \|\varphi\|_{*,\Omega_{\text{PML}}}} \geq C \frac{\min(1, \sigma_0)}{(1 + k^2 R^2)(1 + \sigma_0^2)^{1/2}}.
\]

where $C$ is some generic constant depending only on $\rho/R$ but independent of $R, \rho, \sigma_0$, and

$$
\|\varphi\|_{\Omega^{\text{PML}}} = \left(\|A\nabla \varphi\|_{L^2(\Omega^{\text{PML}})}^2 + \|k\alpha_0 \varphi\|_{L^2(\Omega^{\text{PML}})}^2\right)^{1/2}.
$$

**Proof.** By the definition of the sesquilinear form, it is easy to check that

$$
\Re [c(\varphi, \varphi)] = \int_R \int^\pi_0 2\pi \left(\frac{1 + \sigma \hat{\sigma}}{1 + \sigma^2 r} \left|\frac{1 + \sigma \hat{\sigma}}{1 + \sigma^2 r} \frac{\partial \varphi}{\partial r}\right|^2 + \frac{\sigma - \hat{\sigma}}{1 + \hat{\sigma}^2 r} \left|\frac{\partial \varphi}{\partial \theta}\right|^2 - (1 - \sigma \hat{\sigma}) k^2 r |\varphi|^2\right),
$$

$$
\Im [c(\varphi, \varphi)] = \int_R \int^\pi_0 2\pi \left(\frac{\hat{\sigma} - \sigma}{1 + \hat{\sigma}^2 r} \left|\frac{\partial \varphi}{\partial r}\right|^2 + \frac{\sigma - \hat{\sigma}}{1 + \hat{\sigma}^2 r} \left|\frac{\partial \varphi}{\partial \theta}\right|^2 - (\sigma + \hat{\sigma}) k^2 r |\varphi|^2\right).
$$

Since $\sigma = \sigma_0$ is a constant, we obtain easily

$$
\Re [c(\varphi, \varphi)] - \frac{1}{\sigma_0} \Im [c(\varphi, \varphi)] \geq \int_R \int^\pi_0 2\pi \left(\frac{1 + \sigma \hat{\sigma}}{1 + \sigma^2 r} \left|\frac{\partial \varphi}{\partial r}\right|^2ight) drd\theta,
$$

which implies

$$
\int_R \int^\pi_0 2\pi r \left|\frac{\partial \varphi}{\partial r}\right|^2 drd\theta \leq C \max(1, \sigma_0^{-1})(1 + \sigma_0^2) |c(\varphi, \varphi)|.
$$

Now, since $\varphi = 0$ on $\Gamma_R$, by Poincare inequality,

$$
\int_R \int^\pi_0 2\pi r |\varphi|^2 drd\theta \leq CR^2 \int_R \int^\pi_0 2\pi r \left|\frac{\partial \varphi}{\partial r}\right|^2 drd\theta
\leq CR^2 \max(1, \sigma_0^{-1})(1 + \sigma_0^2) |c(\varphi, \varphi)|.
$$

This completes the proof. \(\square\)

For any $f \in H^{1/2}(\Gamma_R)$, we know that $T f - \hat{T} f = \frac{\partial w}{\partial r} |_{\Gamma_R}$, where $w \in H^1(\Omega^{\text{PML}})$ satisfies

$$
\nabla \cdot (A\nabla w) + \alpha \beta k^2 w = 0 \quad \text{in} \; \Omega^{\text{PML}},
$$

$$
w = 0 \quad \text{on} \; \Gamma_R, \quad w = \mathbb{E}(f) \quad \text{on} \; \Gamma_\rho.
$$

By Lemma 2.1, we have

$$(2.17) \|T f - \hat{T} f\|_{H^{-1/2}(\Gamma_R)} \leq C \max(1, \sigma_0^{-1})(1 + k^2 R^2) \|\mathbb{E}(f)\|_{H^{1/2}(\Gamma_\rho)}.$$

To estimate $\|\mathbb{E}(f)\|_{H^{1/2}(\Gamma_\rho)}$, we recall the following estimate for the first Hankel function in \([11]\) which is proved by Macdonald formula for the product of Hankel functions.

**Lemma 2.2.** For any $\nu \in \mathbb{R}$, $z \in \mathbb{C}_{++} = \{z \in \mathbb{C} : \text{Im} (z) \geq 0, \text{Re} (z) \geq 0\}$, and $\Theta \in \mathbb{R}$ such that $0 < \Theta \leq |z|$, we have

$$
|H^{(1)}_{\nu}(z)| \leq e^{-\text{Im}(z)\left(\frac{1 + \Theta^2}{|z|^2}\right)} \frac{1}{H^{(1)}_{\nu}(\Theta)}.
$$
From Lemma 2.2 we know that

\[(2.18) \quad \| \mathcal{E}(f) \|_{H^{1/2}(\Gamma_r)} \leq e^{-\text{Im}(\rho) \left(1 - \frac{k^2}{\text{Re}(\rho)}\right)^{1/2}} \| f \|_{H^{1/2}(\Gamma_R)}.\]

Combining (2.18) and (2.17), we obtain the exponential decay estimate of \(\| \mathcal{T} f - \hat{T} f \|_{H^{-1/2}(\Gamma_R)}\)

\[(2.19) \quad \| \mathcal{T} f - \hat{T} f \|_{H^{-1/2}(\Gamma_R)} \leq C \max(1, \sigma_0^{-1})(1 + k^2R^2)^2(1 + \sigma_0^2)^3e^{-\text{Im}(\rho) \left(1 - \frac{k^2}{\text{Re}(\rho)}\right)^{1/2}} \| f \|_{H^{1/2}(\Gamma_R)}.\]

2.2. The finite element discretization. Let \(b : H^1(\Omega_\rho) \times H^1(\Omega_\rho) \to \mathbb{C}\) be the sesquilinear form given by

\[
b(\varphi, \psi) = \int_{\Omega_\rho} \left( A \nabla \varphi \cdot \nabla \bar{\psi} - \alpha \beta k^2 \varphi \bar{\psi} \right) dx.
\]

Denote by \(H^1_{(0)}(\Omega_\rho) = \{ v \in H^1(\Omega_\rho) : v = 0 \text{ on } \Gamma_\rho \}\). Then the weak formulation of (2.12a)-(2.12b) is: Given \(g \in L^2(\Gamma_D)\), find \(\hat{u} \in H^1_{(0)}(\Omega_\rho)\) such that

\[(2.20) \quad b(\hat{u}, \psi) = \int_{\Gamma_D} g v \bar{\psi} ds \quad \forall \psi \in H^1_{(0)}(\Omega_\rho).
\]

Let \(\Gamma_h^\rho\), which consists of piecewise segments whose vertices lie on \(\Gamma_\rho\), be an approximation of \(\Gamma_\rho\). Let \(\Omega^\rho_h\) be the subdomain of \(\Omega_\rho\) bounded by \(\Gamma_D\) and \(\Gamma^\rho_h\). Let \(\mathcal{M}_h\) be a regular triangulation of the domain \(\Omega^\rho_h\). We assume the elements \(K \in \mathcal{M}_h\) may have one curved edge align with \(\Gamma_D\) so that \(\Omega^\rho_h = \bigcup_{K \in \mathcal{M}_h} K\). Let \(V_h \subset H^1(\Omega^\rho_h)\) be the conforming linear finite element space over \(\Omega^\rho_h\), and \(\hat{V}_h = \{ v_h \in V_h : v_h = 0 \text{ on } \Gamma^\rho_h \}\). In the following we will always assume that the functions in \(\hat{V}_h\) are extended to the domain \(\Omega_\rho\) by zero so that any function \(v_h \in \hat{V}_h\) is also a function in \(H^1_{(0)}(\Omega_\rho)\). The finite element approximation to the PML problem (2.12a)-(2.12b) reads as follows: Find \(u_h \in \hat{V}_h\) such that

\[(2.21) \quad b(u_h, \psi_h) = \int_{\Gamma_D} g v \bar{\psi}_h ds \quad \forall \psi_h \in \hat{V}_h.
\]

For any \(K \in \mathcal{M}_h\), we denote by \(h_K\) its diameter. Let \(B_h\) denote the set of all sides that do not lie on \(\Gamma_D\) and \(\Gamma^\rho_h\). For any \(e \in B_h\), \(h_e\) stands for its length. For any \(K \in \mathcal{M}_h\), we introduce the residual:

\[
R_h := \nabla \cdot (A \nabla u_h|_K) + \alpha \beta k^2 u_h|_K.
\]

For any interior side \(e \in B_h\) which is the common side of \(K_1\) and \(K_2 \in \mathcal{M}_h\), we define the jump residual across \(e\):

\[
J_e := (A \nabla u_h|_{K_1} - A \nabla u_h|_{K_2}) \cdot \nu_e,
\]
using the convention that the unit normal vector \( \nu \) to \( e \) points from \( K_2 \) to \( K_1 \). If 
\[ e = \Gamma_D \cap \partial K \]
for some element \( K \in M_h \), then we define the jump residual
\[ J_e := 2(\nabla u_h|_K \cdot \mathbf{n} + g) \]
For any \( K \in M_h \), denote by \( \eta_K \) the local error estimator
\[ \eta_K = \left( \| h_K R_h \|^2_{L^2(K)} + \frac{1}{2} \sum_{e \subset \partial K} h_e \| J_e \|^2_{L^2(e)} \right)^{1/2}. \]

**Theorem 2.3.** There exists a constant \( C \) depending only on the minimum angle of the mesh \( M_h \) such that the following a posteriori error estimate is valid
\[ \| u - u_h \|_{H^1(\Omega_R)} \leq C \Lambda(kR)^{1/2}(1 + kR) \left( \sum_{K \in \mathcal{M}_h} \eta_K^2 \right)^{1/2} + C(1 + kR)^2 |\alpha_0|^2 e^{-k \text{Im}(\rho)} \left( 1 + \frac{\rho}{\alpha_0} \right)^{1/2} \| u_h \|_{H^{1/2}(\Gamma_R)}. \]

Here \( \alpha_0 = \max_{\tau \in [R, \rho]} |\alpha(\tau)| \) and \( \Lambda(kR) = \max(1, \frac{\| H^{1/2}(kR) \|}{\| H^{1/2}(kR) \|_0}) \).

For any \( \varphi \in H^1(\Omega_R) \), let \( \tilde{\varphi} \) be its extension in \( \Omega^{\text{PML}} \) such that
\begin{align*}
\nabla \cdot (\bar{A} \nabla \tilde{\varphi}) + \alpha \beta k^2 \tilde{\varphi} &= 0 \quad \text{in} \ \Omega^{\text{PML}}, \\
\tilde{\varphi} &= \varphi \quad \text{on} \ \Gamma_R, \quad \tilde{\varphi} = 0 \quad \text{on} \ \Gamma_\rho.
\end{align*}

The following error representation formula is derived in [11], for any \( \varphi_h \in \tilde{V}_h \),
\[ a(u - u_h, \varphi) = \int_{\Gamma_D} g(\varphi - \varphi_h) - b(u_h, \varphi - \varphi_h) + \langle T u_h - \hat{T} u_h, \varphi \rangle_{\Gamma_R}. \]

The a posteriori error estimate in Theorem 2.3 follows from the error representation formula, the inf-sup condition (2.8), the estimate for \( T - \hat{T} \) in (2.19), and the stability estimate of the extension \( \tilde{\varphi} \).

To derive the stability estimate for \( \tilde{\varphi} \) we consider the function \( w = \tilde{\varphi} - E(\tilde{\varphi}|_{\Gamma_R}) \), then from (2.22)-(2.23) we know that \( w \) satisfies
\begin{align*}
\nabla \cdot (\bar{A} \nabla w) + \alpha \beta k^2 w &= 0 \quad \text{in} \ \Omega^{\text{PML}}, \\
w &= 0 \quad \text{on} \ \Gamma_R, \quad w = -E(\tilde{\varphi}|_{\Gamma_R}) \quad \text{on} \ \Gamma_\rho.
\end{align*}

The stability of the extension \( \tilde{\varphi} \) can be proved by using the stability estimate for the PML extension and the stability estimate of the PML equation in the layer in Lemma 2.1. We refer [11] for the details.

**2.3. The adaptive PML method.** The a posteriori error estimate in Theorem 2.3 is the basis of the adaptive PML method for solving the scattering problem. For the medium property in (2.19) we need only to specify the thickness \( \rho - R \) of the layer and the medium parameter \( \alpha_0 \). From Theorem 2.3 we know that the a posteriori error estimate consists of two parts: the PML error and the finite element
In the adaptive PML method we first choose $\rho$ and $\sigma_0$ such that the exponentially decaying factor:

$$\hat{\omega} = e^{-k\text{Im}(\hat{\rho})(1-\frac{R^2}{|\hat{\rho}|^2})^{1/2}} \leq 10^{-8},$$

which makes the PML error negligible compared with the finite element discretization errors. Once the PML region and the medium property are fixed, we use the standard finite element adaptive strategy to modify the mesh according to the a posteriori error estimate. We refer to [12], [11], and [9] for extensive numerical results of the adaptive PML methods.

In conclusion we remark that the key ingredient in the a posteriori error analysis for the adaptive PML method is to derive the exponential decay estimate of the PML extension which is defined by the complex coordinate stretching of the solution of the original scattering problem outside the PML boundary. For the circular PML boundary, the uniform exponential decay estimate in Lemma 2.2 plays an instrumental role in the analysis. For the rectilinear PML method such as the uniaxial PML method, however, such a method fails and we have to resort to a different technique to derive the required estimate.

### 3. The acoustic problem in layered medium.

In this section we consider the uniaxial PML method for solving the acoustic scattering problems in layered medium

\begin{align}
\Delta u + k(x)^2 u &= 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{D}, \\
u &= g \quad \text{on } \Gamma_D, \\
\sqrt{r} \left( \frac{\partial u}{\partial r} - ik u \right) &\rightarrow 0 \quad \text{as } r = |x| \rightarrow \infty.
\end{align}

Here $D \subset \mathbb{R}^2$ is a bounded domain with Lipschitz boundary $\Gamma_D$ and $g \in H^{1/2}(\Gamma_D)$. We assume the wave number $k$ is positive and piecewise constant, defined by

$$k(x) = \begin{cases} k_1, & \text{if } x \in \mathbb{R}^2_+, \\ k_2, & \text{if } x \in \mathbb{R}^2_-,
\end{cases}$$

where $\mathbb{R}^2_\pm = \{(x_1, x_2) \in \mathbb{R}^2 : \pm x_2 > 0\}$ and $k_2 > k_1 > 0$. Additional continuity conditions are needed across the interface $\Sigma = \{(x_1, 0) : -\infty < x_1 < \infty\}$:

$$[u]_\Sigma = \left[ \frac{\partial u}{\partial x_2} \right]_\Sigma = 0,$$

where $[u]_\Sigma := u_+ - u_-$ is the jump of $u$ across $\Sigma$ from above to below.

We will consider the uniaxial PML method which uses rectilinear PML layers and thus provides greater flexibility and efficiency in dealing with anisotropic scatterers. Let $D$ be contained in the interior of the rectangle $B_1 = \{x \in \mathbb{R}^2 : |x_1| < L_1/2, |x_2| < L_2/2\}$. Let $\Gamma_1 = \partial B_1$ and $\mathbf{n}_1$ the unit outer normal to $\Gamma_1$. We start by introducing the Dirichlet-to-Neumann operator $T : H^{1/2}(\Gamma_1) \rightarrow H^{-1/2}(\Gamma_1)$. Given $f \in H^{1/2}(\Gamma_1)$, we define $Tf = \frac{\partial \xi}{\partial \mathbf{n}_1}$ on $\Gamma_1$, where $\xi$ is the solution of the following
exterior Dirichlet problem of the Helmholtz equation

\begin{align}
\Delta \xi + k(x)^2 \xi &= 0 \quad \text{in } \mathbb{R}^2 \setminus \bar{B}_1, \\
\xi &= f \quad \text{on } \Gamma_1, \\
\sqrt{r} \left( \frac{\partial \xi}{\partial r} - i k \xi \right) &\to 0 \quad \text{as } r = |x| \to \infty.
\end{align}

(3.6) \quad (3.7) \quad (3.8)

It is known that (3.6)-(3.8) has a unique solution \( \xi \in H^1_{\text{loc}}(\mathbb{R}^2 \setminus \bar{B}_1) \) (see e.g. Chen and Zheng [15]). Thus \( T : H^{1/2}(\Gamma_1) \to H^{-1/2}(\Gamma_1) \) is well-defined and is a continuous linear operator.

Fig. 3.1. Setting of the scattering problem with the rectilinear PML layer.

Now we turn to the introduction of the absorbing PML layer. Let \( B_2 = \{ x \in \mathbb{R}^2 : |x_1| < L_1/2 + d_1, |x_2| < L_2/2 + d_2 \} \) be the rectangle which contains \( B_1 \). Let \( \alpha_1(x_1) = 1 + i \sigma_1(x_1), \alpha_2(x_2) = 1 + i \sigma_2(x_2) \) be the model medium property which satisfy \( \sigma_j \in L^\infty(\mathbb{R}), \sigma_j \geq 0, \sigma_j(t) = \sigma_j(-t) \) and \( \sigma_j = 0 \) for \( |t| \leq L_j/2, j = 1, 2 \).

Denote \( \tilde{x}_j \) the complex coordinate

\[
\tilde{x}_j = \begin{cases}
\frac{x_j}{L_j} & \text{if } |x_j| < L_j/2, \\
\int_0^{x_j} \alpha_j(t) dt & \text{if } |x_j| \geq L_j/2.
\end{cases}
\]

To derive the PML equation, we first notice that by the third Green formula, the solution \( \xi \) of the exterior Dirichlet problem (3.6)-(3.8) satisfies

\[ \xi = -\Psi^k_{\text{SL}}(\lambda) + \Psi^k_{\text{DL}}(f) \quad \text{in } \mathbb{R}^2 \setminus \bar{B}_1, \]

where \( \lambda = Tf \in H^{-1/2}(\Gamma_1) \) is the Neumann trace of \( \xi \) on \( \Gamma_1 \), and \( \Psi^k_{\text{SL}}, \Psi^k_{\text{DL}} \) are respectively the single and double layer potentials

\[
\Psi^k_{\text{SL}}(\lambda)(x) = \int_{\Gamma_1} G(x,y) \lambda(y) ds(y), \quad \forall \lambda \in H^{-1/2}(\Gamma_1),
\]

\[
\Psi^k_{\text{DL}}(f)(x) = \int_{\Gamma_1} \frac{\partial G(x,y)}{\partial n_1(y)} f(y) ds(y), \quad \forall f \in H^{1/2}(\Gamma_1).
\]
Here $G$ is the fundamental solution of the Helmholtz equation
\[ \Delta u + k(x)^2 u = 0 \quad \text{in } \mathbb{R}^2 \]
satisfying the Sommerfeld radiation condition \[3.3\]. In the case when $k_1 = k_2$, that is, constant wave number, it is well-known that
\[ G(x, y) = \frac{i}{4} H_0^{(1)}(k|x - y|), \]
where $H_0^{(1)}(z)$ is the first Hankel function of order zero. We know from Lemma \[2.2\] that $H_0^{(1)}(z)$ decays exponentially in the upper half complex plane. In the case of layered medium, the analysis of fundamental solution $G(x, y)$ and its complex stretching is much more completed. We will briefly discuss it in Subsection 3.1 following \[15\].

Now we follow the method of complex coordinate stretching \[17\] to introduce the PML equation. Define
\[ \hat{G}(x, y) = G(\hat{x}, y). \]
It is easy to see that $\hat{G}$ is smooth for $x \in \mathbb{R}^2 \setminus \hat{B}_1$ and $y \in \hat{B}_1$. Define the modified single and double layer potentials
\[ \hat{\Psi}_{SL}^k(\lambda)(x) = \int_{\Gamma_1} \hat{G}(x, y) \lambda(y) \, ds(y), \quad \forall \lambda \in H^{-1/2}(\Gamma_1), \]
\[ \hat{\Psi}_{DL}^k(f)(x) = \int_{\Gamma_1} \frac{\partial \hat{G}(x, y)}{\partial n_1(y)} f(y) \, ds(y), \quad \forall f \in H^{1/2}(\Gamma_1). \]
It is clear that $\hat{\Psi}_{SL}^k(\lambda), \hat{\Psi}_{DL}^k(f)$ are smooth in $\mathbb{R}^2 \setminus \hat{B}_1$, and
\begin{align*}
\gamma_+^D \hat{\Psi}_{SL}^k(\lambda) &= \gamma_+^D \Psi_{SL}^k(\lambda), \quad \forall \lambda \in H^{-1/2}(\Gamma_1), \\
\gamma_+^D \hat{\Psi}_{DL}^k(f) &= \gamma_+^D \Psi_{DL}^k(f), \quad \forall f \in H^{1/2}(\Gamma_1),
\end{align*}
where $\gamma_+^D : H^1_{loc}(\mathbb{R}^2 \setminus \hat{B}_1) \to H^{1/2}(\Gamma_1)$ is the trace operator.

For any $f \in H^{1/2}(\Gamma_1)$, let $\Sigma(f)(x)$ be the PML extension given by
\[ \Sigma(f)(x) = -\hat{\Psi}_{SL}^k(T f) + \hat{\Psi}_{DL}^k(f) \quad \text{for } x \in \mathbb{R}^2 \setminus \hat{B}_1. \]
By \[3.9\] and \[3.10\] we know that $\gamma_+^D \Sigma(f) = -\gamma_+^D \Psi_{SL}^k(T f) + \gamma_+^D \Psi_{DL}^k(f) = \gamma_+^D \xi = f$ on $\Gamma_1$ for any $f \in H^{1/2}(\Gamma_1)$. For the solution $u$ of the scattering problem \[3.1\]-\[3.5\], let $\hat{u} = \Sigma(u|_{\Gamma_1})$ be the PML extension of $u|_{\Gamma_1}$ which satisfies $\gamma_+^D \hat{u} = u|_{\Gamma_1}$ on $\Gamma_1$. We will show in Subsection 3.1 that $\hat{G}$ decays exponentially and thus $u(x)$ will decay exponentially when $x$ is away from $\Gamma_1$. It is obvious that $\hat{u}$ satisfies
\[ \frac{\partial^2 \hat{u}}{\partial x_1^2} + \frac{\partial^2 \hat{u}}{\partial x_2^2} + k(x)^2 \hat{u} = 0 \quad \text{in } \mathbb{R}^2 \setminus \hat{B}_1, \]
which yields the desired PML equation by the chain rule
\[ \nabla \cdot (A \nabla \hat{u}) + \alpha_1 \alpha_2 k(x)^2 \hat{u} = 0 \quad \text{in } \mathbb{R}^2 \setminus \hat{B}_1, \]
where $A = \text{diag}(\alpha_2(x_2)/\alpha_1(x_1), \alpha_1(x_1)/\alpha_2(x_2))$ is a diagonal matrix.

The PML solution $\hat{u}$ in $\Omega_2 = B_2 \setminus \bar{D}$ is defined as the solution of the following system

$$\nabla \cdot (A \nabla \hat{u}) + \alpha_1 \alpha_2 k(x)^2 \hat{u} = 0 \quad \text{in } \Omega_2,$$

$$\hat{u} = g \quad \text{on } \Gamma_D, \quad \hat{u} = 0 \quad \text{on } \Gamma_2.$$

As described in section 2, the convergence of the uniaxial PML method depends on the exponential decay estimate of the PML extension $E(f)$ defined in (3.11) which will be studied in Subsection 3.1 and the stability of the Dirichlet PML problem in the PML layer which we now consider.

Introduce the sesquilinear form $c : H^1(\Omega_{PML}) \times H^1(\Omega_{PML}) \to \mathbb{C}$ as

$$c(\varphi, \psi) = \int_{\Omega_{PML}} (A \nabla \varphi \cdot \nabla \psi - \alpha_1 \alpha_2 k^2 \varphi \psi) \, dx, \quad \forall \varphi, \psi \in H^1_0(\Omega_{PML}).$$

We will use the weighted $H^1$-norm

$$\|\varphi\|_{H^1(\Omega_{PML})} = \left( \|\nabla \varphi\|_{L^2(\Omega_{PML})}^2 + \|k \varphi\|_{L^2(\Omega_{PML})}^2 \right)^{1/2},$$

and the equivalent norm on $H^1(\Omega_{PML})$

$$\|\varphi\|_{*, \Omega_{PML}} = \left( \|A \nabla \varphi\|_{L^2(\Omega_{PML})}^2 + \|k \alpha_1 \alpha_2 \varphi\|_{L^2(\Omega_{PML})}^2 \right)^{1/2}.$$

The following lemma which is similar to Lemma 2.1 is proved in [15].

**Lemma 3.1.** Let $\sigma_j(t) = \sigma, \forall |t| \geq L_j/2, j = 1, 2$, where $\sigma$ is a positive constant. Then it holds that

$$\sup_{0 \neq \varphi \in H^1_0(\Omega_{PML})} \frac{|c(\varphi, \psi)|}{\|\varphi\|_{H^1(\Omega_{PML})}} \geq \hat{C} \|\varphi\|_{*, \Omega_{PML}}, \quad \forall \varphi \in H^1_0(\Omega_{PML}),$$

where

$$\hat{C} = \frac{\min(1, \sigma^3)}{2(1 + \sigma^2)^2 \max(1, k_2^2 d^2)}, \quad d = \max(d_1, d_2).$$

**3.1. The estimate of the PML extension.** We start by studying the Green function for the layered media

$$\Delta_x G(x, y) + k^2 G(x, y) = -\delta_y(x) \quad \text{in } \mathbb{R}^2,$$

$$[G]_\Sigma = \left[ \frac{\partial G}{\partial x_2} \right]_{\Sigma} = 0,$$

where $\delta_y(x)$ is the Dirac source at $y \in \mathbb{R}^2_+$ or $y \in \mathbb{R}^2_-$. Throughout the paper we will always assume that for $z \in \mathbb{C}$, $z^{1/2}$ is the analytic branch of $\sqrt{z}$ such that $\text{Re}(z^{1/2}) \geq 0$. This corresponds to the left half real axis as the branch cut in the complex plane. Denote $\mu_j = (k_j^2 - \xi^2)^{1/2}, j = 1, 2$. Recall
that the Green function for the Helmholtz equation with constant wave number \( k_1 \) is \( \Phi(k_1, x, y) = \frac{i}{4} H_0^{(1)}(k_1|x - y|) \) which satisfies (cf. e.g. [10])

\[
\Phi(k_1, x, y) = \frac{i}{4\pi} \int_{\text{SIP}} \frac{1}{\mu_1 + \mu_2} e^{i(x_1 - y_1)\xi + i\mu_1 |x_2 - y_2|} d\xi.
\]  

(3.12)

Here SIP is the Sommerfeld Integral Path, see Figure 3.1.

By using the method of Fourier transform with respect to the first variable, we can obtain the Green function \( G(x, y) \) for \( x \in \mathbb{R}_+^2, y \in \mathbb{R}_+^2 \),

\[
G(x, y) = \Phi(k_1, x, y) - \Phi(k_1, x, y') + \frac{i}{2\pi} \int_{\text{SIP}} \frac{1}{\mu_1 + \mu_2} e^{i(x_1 - y_1)\xi + i\mu_1 |x_2 - y_2|} d\xi,
\]

where \( y' = (y_1, -y_2) \) is the image of \( y = (y_1, y_2) \), and for \( x \in \mathbb{R}_+^2, y \in \mathbb{R}_+^2 \),

\[
G(x, y) = \frac{i}{2\pi} \int_{\text{SIP}} \frac{1}{\mu_1 + \mu_2} e^{iK(x_1 - y_1) + i\mu_1 |x_2 - y_2|} d\xi.
\]

Similarly we can deduce the Green function for \( x \in \mathbb{R}_+^2, y \in \mathbb{R}_-^2 \),

\[
G(x, y) = \frac{i}{2\pi} \int_{\text{SIP}} \frac{1}{\mu_1 + \mu_2} e^{iK(x_1 - y_1) + i\mu_1 |x_2 - y_2|} d\xi,
\]

and for \( x \in \mathbb{R}_-^2, y \in \mathbb{R}_-^2 \),

\[
G(x, y) = \Phi(k_2, x, y) - \Phi(k_2, x, y') + \frac{i}{2\pi} \int_{\text{SIP}} \frac{1}{\mu_1 + \mu_2} e^{iK(x_1 - y_1) - i\mu_2 |x_2 + y_2|} d\xi.
\]

Let \( h \) be a bounded analytic function in \( \mathbb{C} \setminus ((-\infty, -k_1] \cup [k_1, \infty)) \). For any \( a \in \mathbb{R}, b > 0 \), we denote

\[
I(h; a, b) = \frac{i}{2\pi} \int_{\text{SIP}} \frac{h(\xi)}{\mu_1 + \mu_2} e^{i\xi a + i\mu_1 b} d\xi.
\]  

(3.13)
It is easy to see that the Green function $G(x,y)$ can be represented as follows: for $y \in \mathbb{R}^2$,

$$G(x,y) = \begin{cases} 
\Phi(k_1, x, y) - \Phi(k_1, x, y') + I(1; x_1 - y_1, x_2 + y_2), & \text{if } x_2 > 0, \\
I(e^{i(\mu_1 - \mu_2)x_2}; x_1 - y_1, -x_2 + y_2), & \text{if } x_2 < 0,
\end{cases}$$

and for $y \in \mathbb{R}^2$,

$$G(x,y) = \begin{cases} 
I(e^{i(\mu_1 - \mu_2)y_2}; x_1 - y_1, x_2 - y_2), & \text{if } x_2 > 0, \\
\Phi(k_2, x, y) - \Phi(k_2, x, y') + I(e^{i(\mu_1 - \mu_2)(x_2 + y_2)}; x_1 - y_1, -x_2 - y_2), & \text{if } x_2 < 0.
\end{cases}$$

The following lemma which is proved in [15] by using the method of Cagniard-de Hoop transform plays key role in the estimation of the stretched Green function $\hat{G}$.

**Lemma 3.2.** Let $a \in \mathbb{R}$, $b > 0$, $\rho = \sqrt{a^2 + b^2}$, and $h$ be a bounded analytic function in $\mathbb{C} \setminus ((-\infty, -k_1] \cup [k_1, \infty))$ satisfying $h(\xi) = h(-\xi)$ and $h(\xi) = h(\xi)$. Then

$$I(h; a, b) = \frac{1}{\pi} \int_1^{\infty} \frac{1}{\sqrt{t^2 - 1}} \text{Re} \left[ \left( \frac{\mu_1}{\mu_1 + \mu_2} h \right)(\xi) \right] e^{ik_1\rho t} dt,$$

where $\xi = \frac{k_1|a|t + k_1 b \sqrt{t^2 - 1}}{\rho}$ and $\mu_j = (k_j^2 - \xi^2)^{1/2}$, $j = 1, 2$.

We need the following assumption on the fictitious medium property, which is rather mild in the practical application of the uniaxial PML method.

**H1** \[ \int_0^{L_1 + d_1} \sigma_1(t) dt = \int_0^{L_2 + d_2} \sigma_2(t) dt =: \bar{\sigma}, \quad \bar{\sigma} > 0 \text{ is a constant.} \]

The following estimates for the stretched Green function are proved in [15].

**Lemma 3.3.** Let (H1) and $\gamma_0 \bar{\sigma} \geq \max(k_1^{-1}, \min(d_1, d_2 + L_2/2))$ be satisfied. There exists a constant $C$ depending only on $\gamma_0$, $k_2/k_1$, $L_2/L_1$ but independent of $k_j$, $L_j$, and $d_j$, $j = 1, 2$, such that, for any $x \in \Gamma_2$, $y \in \Gamma_1$, $1 \leq p < 4/3$,

$$|\hat{G}(x,y)| \leq C \gamma_1 e^{-k_1 \gamma_0 \bar{\sigma}},$$

$$\left| \frac{\partial \hat{G}(x,y)}{\partial y_j} \right| \leq C \gamma_1 k_1 \left( 1 + \frac{1}{k_1 L_1} \right) e^{-k_1 \gamma_0 \bar{\sigma}},$$

$$\left| \frac{\partial \hat{G}(x,y)}{\partial x_j} \right| \leq C \gamma_1 k_1 \alpha_m \left( 1 + \frac{1}{k_1 L_1} \right) e^{-k_1 \gamma_0 \bar{\sigma}},$$

$$\left\| \frac{\partial^2 \hat{G}(x,y)}{\partial x_i \partial y_j} \right\|_{L^p(\Gamma_1)} \leq C \gamma_1 k_1^2 L_1^{1/p} \left( 1 + \frac{1}{k_1 L_1} \right)^{2} \left( 1 + \frac{\bar{\sigma}}{L_1} \right)^{2} \alpha_m e^{-k_1 \gamma_0 \bar{\sigma}},$$

where $\alpha_m = \max_{x \in \Omega_{PML}(a_{1}(x_1), a_{2}(x_2))}$, $\gamma_0 := \frac{\min(d_1, d_2 + L_2/2)}{\sqrt{((x_1, d_2 + L_2/2)^2}}} \text{ and } \gamma_1 := e^{L_2 \sqrt{k_2^2 - k_1^2}/2}$. 

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The exponential decay of the PML extension \( E(f) \) in (3.11) can then be proved based on the above lemma. The further details of the convergence analysis and the uniaxial PML method for the layered medium can be found in [15].

4. The electromagnetic scattering problem. In this section we consider the adaptive uniaxial perfectly matched layer (PML) method for solving the time harmonic electromagnetic scattering problem with the perfectly conducting boundary condition

\[
\begin{align*}
\nabla \times \nabla \times \mathbf{E} - k^2 \mathbf{E} &= 0 \quad \text{in } \mathbb{R}^3 \setminus D, \\
\mathbf{n}_D \times \mathbf{E} &= \mathbf{g} \quad \text{on } \Gamma_D, \\
|x| \left[ (\nabla \times \mathbf{E}) \times \hat{x} - i k \mathbf{E} \right] &\to 0 \quad \text{as } |x| \to \infty.
\end{align*}
\]

Here \( D \subset \mathbb{R}^3 \) is a bounded domain with Lipschitz polyhedral boundary \( \Gamma_D \), \( \mathbf{E} \) is the electric field, \( \mathbf{g} \) is determined by the incoming wave, \( \hat{x} = x/|x| \), and \( \mathbf{n}_D \) is the unit outer normal to \( \Gamma_D \). We assume the wave number \( k \in \mathbb{R} \) is a constant.

We first recall some notation. Let \( \Omega \subset \mathbb{R}^3 \) be a Lipschitz domain with boundary \( \Gamma \) whose unit outer normal is denoted by \( \mathbf{n} \). The space

\[ H(\text{curl}; \Omega) = \{ \mathbf{v} \in L^2(\Omega)^3 : \nabla \times \mathbf{v} \in L^2(\Omega)^3 \} \]

is a Hilbert space under the graph norm. The starting point to introduce the traces in \( H(\text{curl}; \Omega) \) is the following Green formula

\[
\int_{\Omega} (\nabla \times \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \nabla \times \mathbf{v}) \, dx = \langle \nabla \times \mathbf{u}, \mathbf{n} \times \mathbf{v} \times \mathbf{n} \rangle_{\Gamma},
\]

for any \( \mathbf{u}, \mathbf{v} \in H^1(\Omega)^3 \), where \( \langle \cdot, \cdot \rangle_{\Gamma} \) is the duality pairing between \( H^{-1/2}(\Gamma)^3 \) and \( H^{1/2}(\Gamma)^3 \). Let \( V_\varepsilon(\Gamma) = \pi_\varepsilon(H^{1/2}(\Gamma)^3) \), where for any \( \mathbf{u} \in H^{1/2}(\Gamma)^3 \), \( \pi_\varepsilon(\mathbf{u}) = \mathbf{n} \times \mathbf{u} \times \mathbf{n} \). We observe from (4.4) that for any \( \mathbf{u} \in H(\text{curl}; \Omega) \), the tangential trace

\[ \gamma_\varepsilon \mathbf{u} = \mathbf{n} \times \mathbf{u} \big|_{\Gamma} \]

can be defined as a continuous linear map on \( V_\varepsilon(\Gamma) \), that is, \( \gamma_\varepsilon \mathbf{u} \in V_\varepsilon(\Gamma)' \). The mapping \( \gamma_\varepsilon : H(\text{curl}; \Omega) \to V_\varepsilon(\Gamma)' \) is, however, not surjective. To define the trace space for \( H(\text{curl}; \Omega) \), we observe that for \( \mathbf{u} \in H(\text{curl}; \Omega) \), \( \nabla \times \mathbf{u} \in H(\text{div}; \Omega) \), and thus \( \nabla \times \mathbf{u} \cdot \mathbf{n} \in H^{-1/2}(\Gamma) \).

Define the surface divergence \( \text{div}_\Gamma: V_\varepsilon(\Gamma)' \to H^{-3/2}(\Gamma) \) as the conjugate operator of the surface gradient \( \nabla_\Gamma : H^{3/2}(\Gamma) \to V_\varepsilon(\Gamma) \). For functions \( \phi \in H^2(\Omega) \), one has \( \nabla_\Gamma \phi = \pi_\varepsilon(\nabla \phi) = \mathbf{n} \times \nabla \phi \times \mathbf{n} \) on \( \Gamma \). Now for any \( \phi \in H^2(\Omega) \),

\[ \langle \text{div}_\Gamma(\mathbf{n} \times \mathbf{u}), \phi \rangle_{3/2, \Gamma} = -\langle \mathbf{n} \times \mathbf{u}, \nabla_\Gamma \phi \rangle_{\Gamma} = -\langle \mathbf{n} \times \mathbf{u}, \mathbf{n} \times \nabla \phi \times \mathbf{n} \rangle_{\Gamma}, \]

where \( \langle \cdot, \cdot \rangle_{3/2, \Gamma} \) is the duality pairing between \( H^{-3/2}(\Gamma)^3 \) and \( H^{3/2}(\Gamma)^3 \). By using (4.3), we obtain

\[ \langle \text{div}_\Gamma(\mathbf{n} \times \mathbf{u}), \phi \rangle_{3/2, \Gamma} = -\int_{\Omega} \nabla \times \mathbf{u} \cdot \nabla \phi \, dx = -\langle \nabla \times \mathbf{u}, \phi \rangle_{\Gamma}. \]
This shows $\text{div}_T(n \times u) = -\nabla \times u \cdot n \in H^{-1/2}(\Gamma)$. Introduce the space
\[ H^{-1/2}(\text{Div}; \Gamma) = \{ \lambda \in V_\pi(\Gamma) : \text{div}_T \lambda \in H^{-1/2}(\Gamma) \}, \]
which is a Hilbert space under the graph norm. Then the tangential trace operator $\gamma_T : H(\text{curl}; \Omega) \to H^{-1/2}(\text{Div}; \Gamma)$ is linear and continuous. That the map $\gamma_T$ is also surjective is established in Buffa et al [7].

Let $D$ be contained in the interior of the domain
\[ B_1 = \{ x = (x_1, x_2, x_3)^T \in \mathbb{R}^3 : |x_i| < L_i/2, \ i = 1, 2, 3 \}. \]

Let $\Gamma_1 = \partial B_1$ and $n_1$ the unit outer normal to $\Gamma_1$. Given a tangential vector $\lambda$ on $\Gamma_1$, the Calderon operator $G_c : H^{-1/2}(\text{Div}; \Gamma_1) \to H^{-1/2}(\text{Div}; \Gamma_1)$ is the Dirichlet-to-Neumann operator defined by
\[ G_c(\lambda) = \frac{1}{i \kappa} n_1 \times (\nabla \times E^s), \]
where $E^s$ satisfies
\begin{align*}
(4.5) \quad & \nabla \times \nabla \times E^s - k^2 E^s = 0 \quad \text{in} \ \mathbb{R}^3 \setminus \bar{B}_1, \\
(4.6) \quad & n_1 \times E^s = \lambda \quad \text{on} \ \Gamma_1, \\
(4.7) \quad & |x| \left[ (\nabla \times E^s) \times \hat{x} - i \kappa E^s \right] \to 0 \quad \text{as} \ |x| \to \infty. 
\end{align*}

Let $a : H(\text{curl}; \Omega_1) \times H(\text{curl}; \Omega_1) \to \mathbb{C}$, where $\Omega_1 = B_1 \setminus \bar{D}$, be the sesquilinear form
\[ a(u, v) = \int_{\Omega_1} (\nabla \times u \cdot \nabla \times v - k^2 u \cdot \nabla v) \, dx + i \kappa (G_c(n_1 \times u), n_1 \times v \times n_1)_{\Gamma_1}. \]

The scattering problem [11, 12, 13] is equivalent to the following weak formulation: Given $g \in H^{-1/2}(\text{Div}; \Gamma_D)$, find $E \in H(\text{curl}; \Omega_1)$ such that $n_D \times E = g$ on $\Gamma_D$, and
\[ a(E, v) = 0, \quad \forall v \in H_D(\text{curl}; \Omega_1), \]
where $H_D(\text{curl}; \Omega_1) = \{ v \in H(\text{curl}; \Omega_1) : n \times v = 0 \text{ on } \Gamma_D \}$.

The existence of a unique solution of the variational problem [4.8] is known [19, 27, 28]. Then the general theory in Babuška and Aziz [2] implies that there exists a constant $\mu > 0$ such that the following inf-sup condition holds
\[ \sup_{v \in H_D(\text{curl}; \Omega_1)} \frac{|a(u, v)|}{\|v\|_{H(\text{curl}; \Omega_1)}} \geq \mu \|u\|_{H(\text{curl}; \Omega_1)}, \quad \forall u \in H_D(\text{curl}; \Omega_1). \]

Now we turn to the introduction of the absorbing PML layer. We assume in the following $d_i \leq CL_i, \ i = 1, 2, 3$, for some generic constant $C$. Let
\[ B_2 = \{ x \in \mathbb{R}^3 : |x_i| < L_i/2 + d_i, \ i = 1, 2, 3 \} \]
be the domain which contains $B_1$. Let $\alpha_i(x_i) = 1 + i \sigma_i(x_i), \ i = 1, 2, 3$, be the model medium property which satisfy
\[ \sigma_i \geq 0, \quad \sigma_i(t) = \sigma_i(-t), \quad \text{and} \quad \sigma_i = 0 \quad \text{for} \ |t| \leq L_i/2, \ i = 1, 2, 3. \]
In order to guarantee the ellipticity of the principal differential operator in the PML equation, one should choose $\sigma_i(t) < 1$. We make the following assumptions on the fictitious medium property $\sigma_i$, which are rather mild in the practical applications of the PML method.

(H2) $\sigma_i(t) < 1$ for $|t| \geq L_i/2$, $i = 1, 2, 3$, and

$$\int_0^{L_i/4} \sigma_1(t) \, dt = \int_0^{L_i/4} \sigma_2(t) \, dt = \int_0^{L_i/4} \sigma_3(t) \, dt =: \bar{\sigma},$$

where $\bar{\sigma} > 0$ is a constant.

Denote $\tilde{x}_i$ the complex coordinate

$$\tilde{x}_i = \begin{cases} x_i & \text{if } |x_i| < L_i/2, \\ \int_0^{L_i/4} \alpha_i(t) \, dt & \text{if } |x_i| \geq L_i/2. \end{cases}$$

To derive the PML equation, we first notice that by the Stratton-Chu integral representation formula, the solution $E^*$ of the exterior Dirichlet problem (3.6)-(3.8) satisfies

$$E^* = \Psi^k_{SL}(\mu) + \Psi^k_{DL}(\lambda) \quad \text{in } \mathbb{R}^3 \setminus \bar{B}_1,$$

where $\mu = G_e(\lambda) \in H^{-1/2}(\text{Div}; \Gamma_1)$ is the Neumann trace of $E^*$ on $\Gamma_1$, and $\Psi^k_{SL}, \Psi^k_{DL}$ are respectively the Maxwell single and double layer potential (cf. e.g. [8])

$$\Psi^k_{SL}(\mu)(x) = ik\Psi^k(\mu)(x) + ik^{-1} \nabla \left[ \Psi^k_{V}(\text{div}_\Gamma, \mu)(x) \right], \quad \forall x \in \mathbb{R}^3 \setminus \bar{B}_1,$$

$$\Psi^k_{DL}(\lambda)(x) = \nabla \times \left[ \Psi^k_{A}(\lambda)(x) \right], \quad \forall x \in \mathbb{R}^3 \setminus \bar{B}_1.$$

Here $\Psi^k_{V}$ and $\Psi^k_{A}$ are the scalar and vector single layer potential for the Helmholtz kernel equation

$$\Psi^k_{V}(\phi)(x) = \int_{\Gamma_1} \phi(y) G(x, y) ds(y), \quad \Psi^k_{A}(\phi)(x) = \int_{\Gamma_1} \phi(y) G(x, y) ds(y)$$
with
\[ G(x, y) = \frac{e^{ik|x-y|}}{4\pi|x-y|}. \]

We follow the method of complex coordinate stretching [17] to introduce the PML equation. Let
\[
\rho(x, y) = \left[ (\tilde{x}_1 - y_1)^2 + (\tilde{x}_2 - y_2)^2 + (\tilde{x}_3 - y_3)^2 \right]^{1/2}
\]
be the complex distance and define
\[
\tilde{G}(x, y) = \frac{e^{ik\rho(x, y)}}{4\pi\rho(x, y)}.
\]

It is easy to see that \( \tilde{G} \) is smooth for \( x \in \mathbb{R}^3 \setminus \tilde{B}_1 \) and \( y \in \tilde{B}_1 \). Define the modified scalar and vector single layer potential for the Helmholtz equation
\[
\tilde{\Psi}_V^k(\phi)(x) = \int_{\Gamma_1} \phi(y) \tilde{G}(x, y) ds(y), \quad \forall \phi \in H^{-1/2}(\Gamma_1),
\]
\[
\tilde{\Psi}_A^k(\phi)(x) = \int_{\Gamma_1} \phi(y) \tilde{G}(x, y) ds(y), \quad \forall \phi \in H^{-1/2}(\Gamma_1)^3,
\]
and the modified single and double layer potential
\[
\tilde{\Psi}_{SL}^k(\mu)(x) = ik \tilde{\Psi}_A^k(\mu)(x) + ik^{-1} \nabla \left[ \tilde{\Psi}_V^k(\text{div}_{\Gamma_1}\mu)(x) \right],
\]
\[
\tilde{\Psi}_{DL}^k(\lambda)(x) = \nabla \times \left[ \tilde{\Psi}_A^k(\lambda)(x) \right].
\]

Here \( \nabla = (\partial/\partial \tilde{x}_1, \partial/\partial \tilde{x}_2, \partial/\partial \tilde{x}_3)^T \) is the gradient operator with respect to the stretched coordinates. It is clear that \( \tilde{\Psi}_{SL}^k(\mu), \tilde{\Psi}_{DL}^k(\lambda) \) are smooth in \( \mathbb{R}^3 \setminus \tilde{B}_1 \), and

(4.12) \[ \gamma_+^+ \tilde{\Psi}_{SL}^k(\mu) = \gamma_+^+ \tilde{\Psi}_{SL}^k(\mu), \quad \forall \mu \in H^{-1/2}(\text{Div}; \Gamma_1), \]
(4.13) \[ \gamma_+^+ \tilde{\Psi}_{DL}^k(\lambda) = \gamma_+^+ \tilde{\Psi}_{DL}^k(\lambda), \quad \forall \lambda \in H^{-1/2}(\text{Div}; \Gamma_1), \]

where \( \gamma_+^+ u = n_1 \times u|_{\Gamma_1} \) for \( u \in H(\text{curl}; \mathbb{R}^3 \setminus \tilde{B}_1) \) is the tangential trace on \( \Gamma_1 \).

For any \( \lambda \in H^{-1/2}(\text{Div}; \Gamma_1) \), let \( \mathcal{E}(\lambda)(x) \) be the PML extension
\[
(4.14) \quad \mathcal{E}(\lambda)(x) = \tilde{\Psi}_{SL}^k(\mu)(x) + \tilde{\Psi}_{DL}^k(\lambda)(x) \quad \text{for} \ x \in \mathbb{R}^3 \setminus \tilde{B}_1,
\]

where \( \mu = G_0(\lambda) \). By (4.12)-(4.13), (4.9), and (4.6) we know that
\[
\gamma_+^+ \mathcal{E}(\lambda) = \gamma_+^+ \tilde{\Psi}_{SL}^k(\mu) + \gamma_+^+ \tilde{\Psi}_{DL}^k(\lambda) = \gamma_+^+ \mathcal{E}^s = \lambda \quad \text{on} \ \Gamma_1.
\]

For the solution \( \mathcal{E} \) of the scattering problem (4.8), let \( \tilde{\mathcal{E}} = \mathcal{E}(n_1 \times \mathcal{E}|_{\Gamma_1}) \) be the PML extension of \( n_1 \times \mathcal{E}|_{\Gamma_1} \). Then \( \gamma_+^+ \tilde{\mathcal{E}} = n_1 \times \mathcal{E}|_{\Gamma_1} \) on \( \Gamma_1 \). It is obvious that \( \tilde{\mathcal{E}} \) satisfies
\[
\tilde{\nabla} \times \tilde{\nabla} \times \tilde{\mathcal{E}} - k^2 \tilde{\mathcal{E}} = 0 \quad \text{in} \ \mathbb{R}^3 \setminus \tilde{B}_1.
\]
It is easy to check that \( \tilde{\nabla} \times \tilde{E} = A \nabla \times (B \tilde{E}) \), where \( A, B \) are diagonal matrices

\[
A = \text{diag} \left( \frac{1}{\alpha_2 \alpha_3}, \frac{1}{\alpha_1 \alpha_3}, \frac{1}{\alpha_1 \alpha_3} \right), \quad B = \text{diag}(\alpha_1, \alpha_2, \alpha_3).
\]

By the chain rule we obtain the desired PML equation

\[
\nabla \times (BA) \nabla \times (B \tilde{E}) - k^2 (BA)^{-1} \tilde{E} = 0 \quad \text{in } \mathbb{R}^3 \setminus B_1.
\]

The PML problem is then to find \( \hat{E} \), which approximates \( E \) in \( \Omega_1 \) and \( B \tilde{E} \) in \( \Omega^{PML} = B_2 \setminus B_1 \), as the solution of the following system

\[
\nabla \times BA \nabla \times \hat{E} - k^2 (BA)^{-1} \hat{E} = 0 \quad \text{in } \Omega_2 = B_2 \setminus \bar{D},
\]

(4.15)

\[
\mathbf{n}_D \times \hat{E} = g \quad \text{on } \Gamma_D, \quad \mathbf{n}_2 \times \hat{E} = 0 \quad \text{on } \Gamma_2.
\]

(4.16)

From the analysis in Section 2 we know the convergence of the PML methods depends on the exponential decay estimate of PML extension \( E(\lambda) \) which we now briefly discuss. The starting point is the following elementary lemma extending a similar result in Chen and Wu [13].

**Lemma 4.1.** For any \( z_i = a_i + ib_i \) with \( a_i, b_i \in \mathbb{R}, i = 1, 2, 3 \), such that \( a_1 b_1 + a_2 b_2 + a_3 b_3 \geq 0 \) and \( a_1^2 + a_2^2 + a_3^2 > 0 \), we have

\[
\text{Im} \left( z_1^2 + z_2^2 + z_3^2 \right)^{1/2} \geq \frac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}.
\]

Now let

\[
z_j = x_j - y_j = (x_j - y_j) + i \int_0^{x_j} \sigma_j(t) \, dt.
\]

For any \( x \in \Gamma_2, y \in \bar{\Omega}_1 \), it is easy to see that \( (x_j - y_j) \int_0^{x_j} \sigma_j(t) \, dt \geq 0 \). Thus, by Lemma 4.1 \( \rho(x, y) = (z_1^2 + z_2^2 + z_3^2)^{1/2} \) satisfies

\[
\text{Im} \rho(x, y) \geq \frac{\sum_{j=1}^3 |x_j - y_j| \int_0^{x_j} \sigma_j(t) \, dt}{\left( \sum_{j=1}^3 |x_j - y_j|^2 \right)^{1/2}}.
\]

Now by (H2) we have

(4.17) \quad \text{Im} \rho(x, y) \geq \gamma \bar{\sigma} \quad \forall x \in \Gamma_2, y \in \bar{\Omega}_1,

where

\[
\gamma = \frac{\min(d_1, d_2, d_3)}{\sqrt{(L_1 + d_1)^2 + (L_2 + d_2)^2 + (L_3 + d_3)^2}}.
\]

We have the following lemma about the decay estimate of the stretched Green function.

**Lemma 4.2.** Let (H2) be satisfied. Let \( \gamma, \bar{\sigma} \) be so chosen that

(4.18) \quad k\gamma \bar{\sigma} \geq 1.
Then there exists a constant $C > 0$ depending only on $\gamma^{-1}$ but independent of $k, \sigma, L_j, d_j$, $j = 1, 2, 3$, such that for any $x \in \Gamma_2, y \in \Omega_1$,

(i) $|\tilde{G}(x, y)| \leq CL^{-1}e^{-\gamma k \hat{\sigma}}$;
(ii) $|\partial G(x, y)/\partial y_j| \leq CK^{-1} e^{-\gamma k \hat{\sigma}}$, $j = 1, 2, 3$;
(iii) $|\partial G(x, y)/\partial x_j| \leq CK^{-1} e^{-\gamma k \hat{\sigma}}$, $j = 1, 2, 3$;
(iv) $|\partial^2 G(x, y)/\partial x_i \partial y_j| \leq CK^2 L^{-1} e^{-\gamma k \hat{\sigma}}$, $i, j = 1, 2, 3$.

Based on Lemma 4.2, the following two lemmas on the estimation the modified single and double layer potentials $\tilde{\Psi}^k_{SL}, \tilde{\Psi}^k_{DL}$ can be proved.

**Lemma 4.3.** Let (H2) and (4.18) be satisfied. For any $\mu \in H^{-1/2}(\text{Div}; \Gamma_1)$, let

$$v(x) = ik \tilde{\Psi}^k_A(\mu)(x) + ik^{-1} \tilde{\nabla} \left[ \tilde{\Psi}^k_V(\text{div}; \mu)(x) \right],$$

be the modified single layer potential. Then

$$\| \mathbf{n}_2 \times Bv \|_{H^{-1/2}(\text{Div}; \Gamma_2)} \leq CKL(1 + kL)^2 e^{-k \gamma \hat{\sigma}} \| \mu \|_{H^{-1/2}(\text{Div}; \Gamma_1)}.$$

**Lemma 4.4.** Let (H2) and (4.18) be satisfied. For any $\lambda \in H^{-1/2}(\text{Div}; \Gamma_1)$, let

$$v(x) = \tilde{\Psi}^k_{DL}(\lambda) = \tilde{\nabla} \times \left[ \tilde{\Psi}^k_A(\lambda)(x) \right],$$

be the double layer potential. Then

$$\| \mathbf{n}_2 \times Bv \|_{H^{-1/2}(\text{Div}; \Gamma_2)} \leq CKL(1 + kL)^2 e^{-k \gamma \hat{\sigma}} \| \lambda \|_{H^{-1/2}(\text{Div}; \Gamma_1)}.$$

Combining lemmas 4.3, 4.4 we have that, for any $\lambda \in H^{-1/2}(\text{Div}; \Gamma_1)$,

$$\| \mathbf{n}_2 \times B\mathbf{E}(\lambda) \|_{H^{-1/2}(\text{Div}; \Gamma_2)} \leq CKL(1 + kL)^2 e^{-k \gamma \hat{\sigma}} \| \lambda \|_{H^{-1/2}(\text{Div}; \Gamma_1)}.$$

We refer to Chen, Cui and Zhang [14] for further details on the a posteriori error analysis and adaptive uniaxial PML method in which extensive numerical experiments are also presented.

**REFERENCES**


