Note on a Nonlinear Evolution Equation for the Risk Preference

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Abstract—We propose a singular nonlinear partial differential equation (PDE), which describes the evolution of the risk preference in the optimal investment problem under the random risk process. The quantity is related to the Arrow-Pratt coefficient of relative risk aversion with respect to the optimal value function. We prove an existence theorem for this PDE.

Keywords: optimal economic behavior, Arrow-Pratt coefficient of relative risk aversion, risk preference, singular nonlinear partial differential equation

1 Introduction

The optimal behaviors in an economic environment have been an intensive subject for researches. Various models have been introduced so far and much progress has been made. We recall that standard way of investigation has been based on stochastic control framework and a number of authors have reduced the analysis to the treatment of the Hamilton-Jacobi-Bellman (HJB) equation for the value function. The resulting nonlinear equations, however, are typically hard to solve; it may be not an exaggeration to say that all that we can do is to merely guess a shape of solutions and manage to arrange the parameters. Of course there are weak approaches to these equations and substantial success in mathematics has been made. However, the notion of weak solutions, namely the viscosity solutions, is a little involved and does not seem to meet the wishes of practitioners. As a result the analysis of HJB equations has certainly stayed as principal difficulties to be surmounted.

In this article we propose a singular nonlinear partial differential equations (PDEs) which is derived from the HJB equation for the value function in the optimal investment problem. Although essential difficulties are equivalent to those expressed by the HJB equation, this derived PDE has rather simple looking from the viewpoint of the theory of nonlinear PDEs. Moreover, the unknown quantity is related to the Arrow-Pratt coefficient of relative risk aversion [7] with respect to the optimal value function. In this sense our introduced PDE may be interpreted as the characteristic equation for the risk structure of the model. We do not insist that our PDE would replace the HJB itself but we at least believe that the study of this PDE is interesting as well as important.

The singular parabolic PDE we are going to deal with is of the form

$$
\frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left\{ (1 + \frac{x^2}{w^2}) \frac{\partial u}{\partial x} - \frac{u^2}{x} - \frac{2u}{x} + (\gamma x + a)u \right\}
+ \frac{u^2}{x^2} - \gamma u - \frac{au}{x}, \quad u = u(x,t)
$$

(1)

where $0 < t < T$ and $x > 0$. Positive constant $\gamma$ denotes the risk-free interest rate and $a$ stands for a positive computable parameter. The unknown function $u$ is related to the Arrow-Pratt coefficient of relative risk aversion [7] for the optimal value function; it is natural to assume that $u$ is positive.

The equation (1) extends our previous study [1], which is concerned with the evolution of the risk preference whose unknown quantity is related to the Arrow-Pratt coefficient of the “absolute” risk aversion. The current equation is formulated on the “relative” risk aversion, which is much popular in the financial economics.

2 Results

First we briefly review our model [1, 5]. It is assumed that there is only one risky stock available for investment, whose price $P_t$ at time $t$ is governed by the stochastic differential equation of Black-Scholes-Merton type [3, 6]

$$
dP_t = P_t(\mu dt + \sigma dW_t^{(1)}), \quad \text{where } \mu \text{ and } \sigma \text{ are constants and } \{W_t^{(1)}\}_{t \geq 0} \text{ is a standard Brownian motion. There are also a risk process and a bond, whose price at time } t \text{ are denoted by } Y_t \text{ and } B_t, \text{ respectively, and assumed to be evolved as } dY_t = \alpha dt + \beta dW_t^{(2)}, \quad dB_t = \gamma B_t dt,
$$

where $\alpha$ and $\beta$ ($\beta > 0$) are constants and $\{W_t^{(2)}\}_{t \geq 0}$ is another standard Brownian motion. The interest rate $\gamma > 0$ ($\mu > \gamma$) is supposed to be constant. It is allowed that these two Brownian motions to be correlated with the correlation coefficient $\rho$ ($0 \leq |\rho| < 1$).

The company invests in the risky stock under an investment policy $f$, where $f = \{f_t\}_{0 \leq t \leq T}$ is a suitable, admissible adapted control process. $T$ stands for the maturity date. Let $X_t^f$ be the wealth of the company at time $t$
with $X_0 = x$, whose evolution process is given by
\[ dX_t^i = f_i dt + \gamma (X_t^i - f_i) dt + dY_t. \]
\[ = (\gamma X_t^i + f_i(\mu - \gamma) + \alpha) dt + f_i \sigma dW_t^{(1)} + dW_t^{(2)}. \]

Suppose that the investor wants to maximize the utility $U(x)$ from his terminal wealth. The utility function $U(x)$ is customarily assumed to satisfy $U' > 0$ and $U'' < 0$. Let
\[ V(x, t) := \sup \int e^{-\delta(T-t)} U(X_T^i) \mid X_t^i = x, \]
where $\delta$ stands for the rate at which consumption and terminal wealth are discounted. We remark that in the seminal work of Browne [2] the case of $\delta = 0$ is treated.

Now the Hamilton-Jacobi-Bellman equation for the value function (2) becomes
\[ \sup \{ A^i V(x, t) \} = -\delta V, \quad V(x, T) = U(x), \] (3)
where the generator $A^i$ is expressed as
\[ (A^i g)(x, t) := \frac{\partial g}{\partial t} + (f(\mu - \gamma) + v - 2\sigma \rho f) \partial_x g + \frac{1}{2} (f^2 \sigma^2 + \beta^2 + 2 \beta \sigma \rho f)^2 \sigma^2 \partial_x^2 g. \]

Suppose that (3) has a classical solution $V$ with $\partial V/\partial x > 0$, $\partial^2 V/\partial x^2 < 0$. We then infer that the optimal policy $\{ f_t^i \}_{0 \leq t \leq T}$ is
\[ f_t^i = -\frac{\mu - \gamma}{\sigma^2} \partial V/\partial x - \frac{\beta \rho}{\sigma}, \] (4)

Placing (4) back into (3) we obtain
\[ \frac{\partial V}{\partial t} + (\gamma x + \gamma x + \alpha - \beta \rho(\mu - \gamma) \frac{\partial V}{\partial x}) \partial V \]
\[ = -\delta V \quad \text{for } 0 < t < T \]
\[ V(T, x) = U(x). \] (5)

Browne [2] shows that if $\delta = 0$ (5) possesses a solution in the case $U(x) = \lambda - (v/\theta) e^{-\delta x}$ with positive constants $\lambda$, $\nu$, $\theta$. This utility has constant absolute risk aversion parameter $\theta$; precisely stated, $-U''(x)/U'(x) = \theta$. Here we proceed further in the analysis of (5).

Let $v(x, t)$ be defined by $V(x, t) = v(E(x, F(T-t)))$, where $E := (\mu - \gamma) / (1 - \rho^2)^{-1/2} \sigma^{-1}$, $F := 2(\mu - \gamma) \sigma^{-2}$. We further define $a := EF^{-1}(\alpha - \beta \rho \sigma^{-1}(\mu - \gamma))$, $b := \delta F^{-1}$, and write $\gamma/F$ by the same $\gamma$ with abuse of notation. It follows that after a calculation
\[ \frac{\partial v}{\partial t} \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \frac{\partial^2 v}{\partial x^2} + (\gamma x + a) \frac{\partial v}{\partial x} + bv, \] (6)

\[ v(x, 0) = U(E^{-1} x). \]

In our previous study [1], we additionally introduce the next quantity.
\[ r(x, t) := \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \log |\frac{\partial v}{\partial x}(x, t)|. \] (7)

It should be noted that (7) is related to the coefficient of absolute risk aversion. A little tedious computation then leads us to the following equation.
\[ \frac{\partial r}{\partial t} = \frac{\partial}{\partial x} \left( (1 + \frac{1}{\tau^2}) \frac{\partial r}{\partial x} - \tau^2 + (\gamma x + a) \right), \] (8)
\[ r = r(x, t) \quad \text{in } (x, t) \in \Omega_T := R^+ \times (0, T) \]
where $T > 0$ and $R^+ = \{ x > 0 \}$.

Compared to the equation (6), the equation (8) is quasilinear and has divergence form; (8) is rather popular type in the PDE world, although it is singular at the same time.

The current equation (1) is now deduced if we define
\[ u(x, t) := x r(x, t) = -x \frac{\partial^2 v}{\partial x^2} - \frac{\partial v}{\partial x} \log |\frac{\partial v}{\partial x}(x, t)|. \] (9)

Here we consider the equation (1) on the bounded interval $[L_0, L_1] \subset R^+$ with the Dirichlet boundary condition
\[ u(L_0, t) = a_0, \quad u(L_1, t) = a_1, \] (10)
where $a_0, a_1$ are positive constants.

We now address the next existence theorem, whose proof is given along the same line of Ishimura and Murao [5] and we may safely omit the details.

**Theorem.** For any initial data $u_0(x) \in C([L_0, L_1])$ which satisfies the compatibility condition (10), there corresponds $T = T(u_0)$ such that the solution $u$ of (1) which verifies (10) exists on $[L_0, L_1] \times (0, T)$.

**References**


