An implicit numerical scheme for the modified Burgers’ equation

A. G. Bratsos
Department of Mathematics,
Technological Educational Institution (T.E.I.) of Athens,
GR-122 10 Egaleo, Athens, Greece.

Abstract
A finite-difference scheme based on rational approximants to the matrix-exponential term in a two-time level recurrence relation has been proposed for the numerical solution of the modified Burgers’ equation already known in the bibliography. The method leads to the solution of a nonlinear system. To overcome this difficulty a predictor-corrector scheme is proposed. The efficiency of the proposed method is tested to a known from the bibliography problem.

Keywords
Burgers’ equation; Modified Burgers’ equation; Finite-difference method; Predictor-Corrector.
AMS — 35Q53; 65M06; 78M20.

I. INTRODUCTION

BURGERS [1], [2] using earlier studies in [3] introduced an equation to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion. Since then Burgers’ equation was found to be a fundamental equation in fluid mechanics. It occurs in various areas of applied mathematics, such as in modelling of gas dynamics and traffic flow (see for extended details in [8]).

The modified Burgers’ equation (MBE) has in general the form

\[ u_t + uu_x - \nu u_{xx} = 0; \quad L_0 < x < L_1, \quad t > t_0, \]  

where \( \mu \) is a positive integer with \( \mu \geq 2 \) - the case \( \mu = 1 \) corresponds to the classical Burgers’ equation, \( u = u(x, t) \) is a sufficiently often differentiable function and \( \nu \) is a constant, which can be interpreted as viscosity, controlling the balance between convection and diffusion. The MBE equation has the strong nonlinear aspects of the governing equation in many practical transport problems such as nonlinear waves in a medium with low-frequency pumping or absorption, ion reflection at quasi-perpendicular shocks, turbulence transport, wave processes in thermoelastic medium, transport and dispersion of pollutants in rivers and sediment transport, etc. Numerical solutions of the MBE equation were found among others in [4]-[11].

The initial condition associated with Eq. (1.1) will be

\[ u(x, t_0) = f(x); \quad L_0 \leq x \leq L_1, \]  

while the boundary conditions

\[ u(L_i, t); \quad i = 0, 1 \quad \text{and} \quad u_x|_{x=L_i} = 0; \quad i = 0, 1; \quad t > t_0. \]  

II. THE NUMERICAL METHOD

A. Development of the method

To obtain numerical solutions the region \( R = [L_0 < x < L_1] \times [t > t_0] \) with its boundary \( \partial R \) consisting of the lines \( x = L_0, \quad x = L_1 \) and \( t = t_0 \) is covered with a rectangular mesh, \( G \), of points with co-ordinates \( (x, t) = (x_m, t_n) \) = \( (L_0 + mh, t_0 + n\ell) \) with \( m = 0, 1, \ldots, N + 1 \). The theoretical solution of Eq. (1.1) at this typical mesh point \( (x_m, t_n) \) will be denoted by \( u^n_m \) and the relevant of an approximating difference scheme by \( U^n_m \).

Let the solution vector at time \( t = t_n = t_0 + n\ell \) be

\[ U^n = U(t_n) = [U^n_0, U^n_1, \ldots, U^n_{N+1}]^T. \]  

E-mail: bratsos@teiath.gr
URL: http://math.teiath.gr/bratsos
tridiagonal matrices and µ where D

The proposed method

Replacing the space derivatives with the familiar central-difference formulas and applying Eq. (1.1) at each point of the grid G at time level \( t = t_0 + n\ell; n = 0, 1, \ldots \) leads to a first-order initial-value problem, which is written in a matrix-vector form as

\[
D \mathbf{U}(t) = -\text{diag}\{((U_m^n)^\mu}\} A \mathbf{U}(t) + \nu B \mathbf{U}(t)
\]

\[
\mathbf{U}(0) = \mathbf{f}; \quad t > t_0
\] (2.2)
in which \( D = \{d/dt\} \) is a diagonal matrix of order \( N + 2 \) and

\[
A = \frac{1}{2h} \begin{bmatrix}
-2 & 2 \\
-1 & 0 & 1 \\
& \ddots & \ddots & \ddots \\
& & -1 & 0 & 1 \\
& & & 2 & -2
\end{bmatrix}, \quad B = \frac{1}{h^2} \begin{bmatrix}
-2 & 2 \\
1 & -2 & 1 \\
& \ddots & \ddots & \ddots \\
& & 1 & -2 & 1 \\
& & & 2 & -2
\end{bmatrix}
\] (2.3)

tridiagonal matrices and \( \mathbf{f} \) the vector of the initial condition all of order \( N + 2 \).

Relation (2.2) gives

\[
D = -\text{diag}\{((U_m^n)^\mu}\} A + \nu B,
\] (2.4)

so

\[
D^2 = (\text{diag}\{((U_m^n)^\mu}\} A)^2 - \nu (\text{diag}\{((U_m^n)^\mu}\} A B + B \text{diag}\{((U_m^n)^\mu}\} A) + \nu^2 B^2.
\] (2.5)

The numerical methods will be developed by replacing the matrix-exponential term in the recurrence relation

\[
\mathbf{U}(t + \ell) = \exp(\ell D) \mathbf{U}(t); \quad t = t_0 + \ell, t_0 + 2\ell, \ldots
\] (2.6)

where \( D \mathbf{U}(t) \) given by (2.2), by rational replacements, which are also known as the \((\mu, \nu)\) Padé approximants of order \( \mu + \nu \) to \( \exp(\ell D) \) [12], of the form \( \exp(\ell D) = (I + a_1 \ell D + b_1 \ell^2 D^2)^{-1} (I + c_1 \ell D + d_1 \ell^2 D^2) \) in which \( a_1, b_1, c_1, d_1 \) are real parameters having appropriate values for each type of the approximants given in Table I. The expression of Eq. (2.6) arising from the use of these parameters for the methods, which are going to be examined in this paper, is given in Table II.

### B. The proposed method

The method arises from Method II in Table II which subject to (2.4)-(2.5) leads to the following nonlinear system

\[
\mathbf{U}(t + \ell) - \frac{1}{2} \ell \left[-\text{diag}\{((U_m^n+1)^\mu}\} A + \nu B\right] \mathbf{U}(t + \ell) + \frac{1}{12} \ell^2 \left[(\text{diag}\{((U_m^n+1)^\mu}\} A)^2 + \nu^2 B^2\right]
\]

\[
-\nu \left[(\text{diag}\{((U_m^n+1)^\mu}\} A B + B \text{diag}\{((U_m^n+1)^\mu}\} A)\right] \mathbf{U}(t + \ell) = \mathbf{U}(t) + \frac{1}{2} \ell \left[-\text{diag}\{((U_m^n)^\mu}\} A + \nu B\right] \mathbf{U}(t)
\]

\[
+ \frac{1}{12} \ell^2 \left[(\text{diag}\{((U_m^n)^\mu}\} A)^2 + \nu^2 B^2\right] - \nu (\text{diag}\{((U_m^n)^\mu}\} A B + B \text{diag}\{((U_m^n)^\mu}\} A) \mathbf{U}(t)
\] (2.7)

Let \( r_1 = \ell \nu / 2h^2, r_2 = \ell / 4h, r_3 = \ell^2 / 48h^2, r_4 = \ell^2 \nu / 24h^3 \) and \( r_5 = \ell^2 \nu^2 / 12h^4 \). Eq. (2.7), when applied to the general mesh point of the grid G, gives

\[
U_{m+1}^{n+1} - r_1 \left(U_{m-1}^{n+1} - 2U_m^{n+1} + U_{m+1}^{n+1}\right) = r_2 \left(U_{m-1}^{n+1} - U_{m+1}^{n+1}\right) \left(U_m^{n+1}\right)^\mu
\]
\[ +r_3 \left\{ (U_{m-1}^{n+1})^\mu U_{m-2}^{n+1} - \left[ (U_{m-1}^{n+1})^\mu + (U_{m+1}^{n+1})^\mu \right] (U_{m}^{n+1})^\mu + (U_{m+1}^{n+1})^\mu (U_{m+2}^{n+1})^\mu \right\} \\
- r_4 \left\{ 4 (U_{m}^{n+1})^\mu U_{m-1}^{n+1} - \left[ (U_{m-1}^{n+1})^\mu + (U_{m+1}^{n+1})^\mu \right] U_{m-2}^{n+1} - 4 (U_{m}^{n+1})^\mu U_{m+1}^{n+1} + \left[ (U_{m-1}^{n+1})^\mu - (U_{m+1}^{n+1})^\mu \right] U_{m+2}^{n+1} \right\} \\
+ \left\{ (U_{m}^{n+1})^\mu + (U_{m+1}^{n+1})^\mu \right\} U_{m+2}^{n+1} + r_5 \left\{ (U_{m-2}^{n+1} - 4U_{m-1}^{n+1} + 6U_{m}^{n+1} - 4U_{m+1}^{n+1} + U_{m+2}^{n+1}) \right\} \quad m = 0, 1, \ldots, N + 1. \quad (2.8) \]

C. The Predictor-Corrector scheme

To avoid solving the nonlinear system (2.7) the following Predictor-Corrector (P-C) scheme is proposed.

C.1 Predictor

The value \( \hat{U}(t + \ell) \) is evaluated from Method I, whose relevant expression in Table II subject to Eq. (2.4) is written as [11]

\[ \hat{U}(t + \ell) = \left( I + \ell D + \frac{\ell^2}{2} D^2 \right) U(t) \]

otherwise using (2.4)-(2.5)

\[ \hat{U}(t + \ell) = U(t) + \ell (-\text{diag} \{ U_n^m \}) A + \nu B) U(t) + \frac{1}{2} \ell^2 \left[ \text{diag} \{ U_n^m \} A^2 + \nu^2 B^2 \right] - \nu (\text{diag} \{ U_n^m \}) A B + B \text{diag} \{ U_n^m \} A) U(t). \quad (2.9) \]

Let \( p_1 = \ell \nu/h^2 \), \( p_2 = \ell/2 \), \( p_3 = \ell^2/8 h^2 \), \( p_4 = \ell^2 \nu/4 h^3 \) and \( p_5 = \ell^2 \nu^2/2 h^4 \). Eq. (2.9), when applied to the general mesh point of the grid \( G \), gives

\[ U_{m+1}^{n+1} = U_m^n + p_1 \left( U_{m-1} - 2U_m^n + U_{m+1} \right) + p_2 \left( U_{m-1} - U_{m+1} \right) \]

\[ + p_3 \left[ U_{m-2} U_{m-1} - U_m^n \left( U_{m-1} + U_{m+1} \right) \right] + p_4 \left( U_{m-2} U_{m-1} + U_{m+1} \right) \]

\[ + p_5 \left( U_{m-2} - 4U_{m-1} + 6U_m^n - 4U_{m+1} + U_{m+2} \right) ; \]

\( m = 0, 1, \ldots, N + 1. \quad (2.10) \)

C.2 Corrector

The corrector arises from Method II in Table II as follows

\[ \hat{U}(t + \ell) = \left( \frac{1}{2} \ell D - \frac{1}{12} \ell^2 D^2 \right) \hat{U}(t + \ell) + \left( I + \frac{1}{2} \ell D + \frac{1}{12} \ell^2 D^2 \right) U(t). \quad (2.11) \]

III. Numerical results

For the linearization \( U_0 = \max_{m=0,1,\ldots,N+1} \{ u_t^m \} \) was given. Let \( e_m^n = | u_m^n - U_0 |; m = 0, 1, \ldots, N + 1 \). Then the error at time level \( t = t_0 + n \ell; n = 0, 1, \ldots, e = L_{\infty} = \max_{m=0,1,\ldots,N+1} e_m^n \) and the corresponding error \( L_2 = \sqrt{h \sum_{m=0}^{N+1} (e_m^n)^2} \).

Following [13] MBE equation has the analytic solution

\[ u(x, t) = \frac{x}{t} \left[ \frac{1}{t} \sqrt{\frac{\sqrt{t}}{t_0}} \exp \left( \frac{x^2}{4 \nu t} \right) \right]^{-1} , \quad 0 \leq x \leq 1; \ t \geq 1, \quad (3.1) \]

with \( t_0 \in (0, 1) \). The initial condition \( u(x, 1) \) is given from Eq. (3.1), while the boundary conditions (1.3) are \( g_0(t) = 0 \), \( g_0(t) = u(1, t) \) with \( u(1, t) \) given by Eq. (3.1) and \( u_{x=0} u_{x=1} = 0 \).

BME equation was solved for various values of the space and time steps using \( \nu = 0.005, 0.002 \). From the experiments the most accurate results were obtained for \( h = 0.005 \) and \( \ell = 10^{-5} \) - see Table I and Fig. 1, while for \( \ell < 10^{-5} \) no increment in the accuracy occurred.
TABLE III
MBE equation. Numerical results using $h = 0.005$ and $\ell = 10^{-5}$.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$L_\infty \times 10^3$</th>
<th>$L_2 \times 10^3$</th>
<th>$L_\infty \times 10^3$</th>
<th>$L_2 \times 10^3$</th>
<th>$L_\infty \times 10^4$</th>
<th>$L_2 \times 10^4$</th>
<th>$L_\infty \times 10^5$</th>
<th>$L_2 \times 10^5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.005</td>
<td>0.58090</td>
<td>0.22711</td>
<td>0.42972</td>
<td>0.18829</td>
<td>0.33000</td>
<td>0.16461</td>
<td>0.22873</td>
<td>0.13521</td>
</tr>
<tr>
<td>0.002</td>
<td>0.36923</td>
<td>0.11510</td>
<td>0.27279</td>
<td>0.09507</td>
<td>0.20931</td>
<td>0.08302</td>
<td>0.14501</td>
<td>0.06830</td>
</tr>
</tbody>
</table>

Fig. 1. MBE equation: The dashed curve shows the theoretical solution $u$ at $t = 1$ (initial condition) and the full curves the numerical solution $U$ at $t = 2, 4, 6$ and 10 (lowest curve) when $\nu = 0.005$.

IV. Conclusions

An implicit finite-difference scheme based on fourth order rational approximants to the matrix-exponential term was proposed for the numerical solution of the modified Burgers’ equation. The resulting nonlinear scheme was solved using an appropriate predictor-corrector scheme. Numerical results examining the efficiency of the proposed method were derived.

References