An implicit numerical scheme for the modified Burgers’ equation

A. G. Bratsos
Department of Mathematics,
Technological Educational Institution (T.E.I.) of Athens,
122 10 Egaleo, Athens, Greece.

Abstract
An implicit finite-difference scheme based on rational approximants to the matrix-exponential term in a two-time level recurrence relation is proposed for the numerical solution of the modified Burgers’ equation already known from the bibliography. The resulting system is solved using a predictor-corrector scheme. The efficiency of the proposed method is tested to a known problem.

Keywords
Burgers’ equation; Modified Burgers’ equation, Finite-difference method; Predictor-Corrector.
AMS — 35Q53; 65M06; 78M20.

I. Introduction
Burgers’ \cite{1}, \cite{2} using earlier studies in \cite{3} introduced an equation to capture some of the features of turbulent fluid in a channel caused by the interaction of the opposite effects of convection and diffusion. Since then Burgers’ equation was found to be a fundamental equation in fluid mechanics. It occurs in various areas of applied mathematics, such as modelling of gas dynamics and traffic flow (see for extended details in \cite{8}).

The modified Burgers’ equation (MBE) has in general the form
\begin{equation}
  u_t + u^\mu u_x - \nu u_{xx} = 0; \quad L_0 < x < L_1, \quad t > t_0,
\end{equation}
where $\mu$ is a positive integer with $\mu \geq 2$ - the case $\mu = 1$ corresponds to the classical Burgers’ equation, $u = u(x, t)$ is a sufficiently often differentiable function and $\nu$ is a constant, which can be interpreted as viscosity, controlling the balance between convection and diffusion. The MBE equation has the strong nonlinear aspects of the governing equation in many practical transport problems such as nonlinear waves in a medium with low-frequency pumping or absorption, ion reflection at quasi-perpendicular shocks, turbulence transport, wave processes in thermoelastic medium, transport and dispersion of pollutants in rivers and sediment transport, etc. Numerical solutions of the MBE equation were found among others in \cite{4}-\cite{10}.

The initial condition associated with Eq. (1) will be
\begin{equation}
  u(x, t_0) = f(x); \quad L_0 \leq x \leq L_1,
\end{equation}
while the boundary conditions
\begin{equation}
  u(L_i, t) = g_i(t) ; \quad i = 0, 1 \quad \text{and} \quad u_x|_{x=L_i} = 0 ; \quad i = 0, 1; \quad t > t_0.
\end{equation}

II. The numerical method
To obtain numerical solutions the region $R = [L_0 < x < L_1] \times [t > t_0]$ with its boundary $\partial R$ consisting of the lines $x = L_0$, $x = L_1$ and $t = t_0$ is covered with a rectangular mesh, $G$, of points with co-ordinates $(x, t) = (x_m, t_n)$ $=$ $(L_0 + mh, t_0 + n\ell)$ with $m = 0, 1, ..., N + 1$. The theoretical solution of Eq. (1) at the typical mesh point $(x_m, t_n)$ will be denoted by $u^n_m$, while the relevant of an approximating difference scheme by $U^n_m$.

Let the solution vector at time $t = t_n = t_0 + n\ell$ be
\begin{equation}
  \mathbf{U}^n = U(t_n) = [U^n_0, U^n_1, ..., U^n_{N+1}]^\top.
\end{equation}
Replacing the space derivatives with the familiar central-difference formulas and applying Eq. (1) at each point of the grid $G$ at time level $t = t_0 + n\ell$; $n = 0, 1, ...$ leads to a first-order initial-value problem, which is written in a matrix-vector form as
\begin{equation}
  D \mathbf{U}(t) = -\text{diag} \{((u^n_m)^\mu)\} \ A \mathbf{U}(t) + \nu \mathbf{B} \mathbf{U}(t)
\end{equation}
\begin{equation}
  \mathbf{U}(0) = \mathbf{f}; \quad t > t_0
\end{equation}
E-mail: bratsos@teiath.gr
URL: http://math.teiath.gr/bratsos
TABLE I

<table>
<thead>
<tr>
<th>Method</th>
<th>$(\mu, \nu)$ Pade</th>
<th>Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$(0,2)$</td>
<td>$U(t + \ell) = \left( I + \ell D + \frac{\ell^2 D^2}{2} \right) U(t) \quad (6)$</td>
</tr>
<tr>
<td>II</td>
<td>$(2,2)$</td>
<td>$\left( I - \frac{1}{2} \ell D + \frac{1}{12} \ell^2 D^2 \right) U(t + \ell) = \left( I + \frac{1}{2} \ell D + \frac{1}{12} \ell^2 D^2 \right) U(t)$</td>
</tr>
</tbody>
</table>

in which $D = \{d/dt\}$ is a diagonal matrix, $A, B$ tridiagonal matrices with appropriate entries and $f$ the vector of the initial condition all matrices of order $N + 2$.

Numerical methods will be developed by replacing the matrix-exponential term in the recurrence relation

$$U(t + \ell) = \exp(\ell D) U(t) \quad ; \; t = t_0 + \ell, t_0 + 2\ell, ...$$

where $D U(t)$ is given by (5), by rational replacements [11]. The expression of the methods, which are going to be examined, is given in Table I.

The nonlinear method arises from Method II in Table I. To avoid solving the resulting nonlinear system a Predictor-Corrector scheme using Method I as Predictor is proposed.

III. Numerical results

The proposed P–C scheme was tested numerically to the problem introduced by [12] with analytical solution

$$u(x, t) = \frac{x}{t} \left[ 1 + \frac{\sqrt{t}}{t_0} \exp \left( \frac{x^2}{4\nu t} \right) \right]^{-1}, \; 0 \leq x \leq 1 ; \; t_0 \in (0, 1) \text{ with } t \geq 1$$

and numerical results at various time levels were derived.

References