A linearly implicit finite element method for a Klein–Gordon–Schrödinger-type system

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Abstract

We consider a Klein–Gordon–Schrödinger-type system of equations in one space dimension, that describes the nonlinear interaction between high frequency electron waves and low frequency ion plasma waves in a homogeneous magnetic field. To approximate its solution, we propose a linearly implicit finite element method, which is unconditionally well-defined and achieves optimal order convergence in the L^2 and H^1 norms.

Index Terms

linearly implicit time stepping, finite element method, a priori error estimates

I. Introduction

A. Formulation of the problem

Let $T>0$, $\Omega:=(x_{A},x_{B})\subset\mathbb{R}$ be a bounded interval and $D:=[0,T]\times\overline{\Omega}$. Then, we consider the following initial- and boundary- value problem for a dissipative Klein–Gordon–Schrödinger (DKGS) system of partial differential equations: find functions $\psi:D\rightarrow\mathbb{C}$ and $\phi:D\rightarrow\mathbb{R}$ solving the following system of equations

$$\psi_t = i\,\mu\,\psi_{xx}-\alpha\,\psi-i\,\phi\,\psi \quad \text{on} \quad (0,T]\times\Omega, \tag{I.1}$$

$$\phi_{tt} = \phi_{xx}-\phi-\lambda\,\phi_t+\mathcal{L}\psi \quad \text{on} \quad (0,T]\times\Omega, \tag{I.2}$$

with

$$\mathcal{L}\psi := -\mathrm{Re}(\psi_x), \tag{I.3}$$

and satisfying the conditions

$$\psi(t,x)=0 \quad \text{and} \quad \phi(t,x)=0 \quad \forall\,(t,x)\in[0,T]\times\partial\Omega, \tag{I.4}$$

$$\psi(0,x)=\psi_0(x) \quad \forall\,x\in\overline{\Omega}, \tag{I.5}$$

$$\phi(0,x)=\phi_0(x) \quad \text{and} \quad \phi_t(0,x)=\phi_1(x) \quad \forall\,x\in\overline{\Omega}, \tag{I.6}$$

where: μ, α and λ are known real positive numbers, and ϕ_0, $\phi_1:\overline{\Omega}\rightarrow\mathbb{R}$, $\psi_0:\overline{\Omega}\rightarrow\mathbb{C}$ are known smooth functions. The DKGS system models the nonlinear interaction between high frequency electron waves and low frequency ion plasma waves in a homogeneous magnetic field. Such systems arise in the UHH plasma heating technique. The coupling of the two fluids is achieved through the (non-homogeneous) polarization drift and the induced current takes part in the collisional process of energy exchange. As a consequence, the nonlinearity differs from

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the one encountered in the Zakharov system [6]. For a detailed derivation of the model, along with the underlying assumptions, the reader is referred to [2] and [5]. Also, we refer to [2] for results on the existence and uniqueness of a solution of the problem. In the present work, we focus on the derivation of a finite element method to approximate the solution of the initial and boundary value problem (I.1)-(I.6).

B. Function spaces and operators

For $p > 1$, we will denote by $L^p(\Omega)$ the space of the Lebesgue measurable complex valued functions which have the $p-$th power of its absolute value integrable on Ω, and by $\|\!|\cdot|\!\|_p$ the standard norm of $L^p(\Omega)$, i.e., $\|\!|f|\!\|_p := \{\int_\Omega |f(x)|^p\,dx\}^{\frac{1}{p}}$ for $f \in L^p(\Omega)$. The inner product in $L^2(\Omega)$ that derives the norm $\|\cdot\|_0 := \|\!|\cdot|\!\|_2$ will be denoted by $(\cdot,\cdot)_0$, i.e. $(f_1,f_2)_0 := \int_\Omega f_1(x)\,\overline{f_2(x)}\,dx$ for f_1, $f_2 \in L^2(\Omega)$. Also, we will denote by $L^\infty(\Omega)$ the space of the Lebesgue measurable complex valued functions which are bounded a.e. on Ω, and by $|\cdot|_\infty$ the standard norm of $L^\infty(\Omega)$, i.e., $|f|_\infty := \operatorname{ess\,sup}_\Omega |f|$ for $f \in L^\infty(\Omega)$. For $s \in \mathbb{N}_0$, we denote by $H^s(\Omega)$ the Sobolev space of complex valued functions having generalized derivatives up to order s in $L^2(\Omega)$, and by $\|\cdot\|_s$ its usual norm, i.e. $\|f\|_s := \left\{\sum_{\ell=0}^s \|\partial^\ell f\|_0^2\right\}^{\frac{1}{2}}$ for $f \in H^s(\Omega)$. In addition, we set $|v|_1 := \|v'\|_0$ for $v \in H^1(\Omega)$. Also, $H_0^1(\Omega)$ will denote the subspace of $H^1(\Omega)$ consisting of functions which vanish at the endpoints of Ω in the sense of trace. For $s \in \mathbb{N}_0$, we denote by $W^{s,\infty}(\Omega)$ the Sobolev space of complex valued functions having generalized derivatives up to order s in $L^\infty(\Omega)$, and by $|\cdot|_{s,\infty}$ its usual norm, i.e. $|f|_{s,\infty} := \max_{0\le\ell\le s} |\partial^\ell f|_\infty$ for $f \in W^{s,\infty}(\Omega)$.

For $r \in \mathbb{N}$, let $S_h \subset H_0^1(\Omega)$ be a finite element space consisting of functions which are piecewise polynomials of degree at most r over a partition of Ω in intervals with maximum mesh-length h. It is well-known (cf., e.g., [1]) that there exists a constant $C_0^r > 0$ such that

$$\inf_{\chi\in S_h} \left\{\|v-\chi\|_0 + h\,\|v-\chi\|_1\right\} \le C_0^r\,h^\ell\,\|v\|_\ell \quad \forall\, v \in H^\ell(\Omega)\cap H_0^1(\Omega), \quad \ell = 1,\dots,r+1. \tag{I.7}$$

Now, we define the discrete Laplacian operator $\Delta_h : H^1(\Omega) \to S_h$ by $(\Delta_h\varphi,\chi)_0 = (\varphi',\chi')_0$ for $\varphi,\chi \in S_h$, and the L^2-projection operator $P_h : L^2(D) \to S_h$ by $(P_h f,\chi)_0 = (f,\chi)_0$ for $\chi \in S_h$ and $f \in L^2(D)$. Also, we introduce elliptic projection operators R_h^Ψ, $R_h^\Phi : H^1(\Omega) \to S_h$ by $((R_h^\Psi v)',\chi')_0 = (v',\chi')_0$ for $\chi \in S_h$ and $v \in H^1(\Omega)$, and $((R_h^\Phi v)',\chi')_0 + \lambda\,(R_h^\Phi v,\chi)_0 = (v',\chi')_0 + \lambda\,(v,\chi)_0$ for $\chi \in S_h$ and $v \in H^1(\Omega)$. For $R_h = R_h^\Psi$ or R_h^Φ, it is well known (see [3]) that

$$\|R_h v - \chi\|_0 + h\,\|R_h v - \chi\|_1 \le C_E\,h^\ell\,\|v\|_\ell \quad \forall\, v \in H^\ell(\Omega)\cap H_0^1(\Omega), \quad \ell = 1,\dots,r+1. \tag{I.8}$$

Also, from [4], we have

$$|R_h v - v|_\infty \le C_{E,\infty}\,h^\ell\,|v|_{\ell,\infty} \quad \forall\, v \in W^{\ell,\infty}(\Omega)\cap H_0^1(\Omega), \quad \ell = 1,\dots,r+1. \tag{I.9}$$

II. The Finite Element Method

A. Formulation of the method

Let $N \in \mathbb{N}$ and $k := \frac{T}{N}$. Then, we define $t^m := m\,k$, $\psi^m := \psi(t^m,\cdot)$ and $\phi^m := \phi(t^m,\cdot)$ for $m = 0,\dots,N$. For $m = 0,\dots,N$, the proposed method constructs, an approximation $(\Psi_h^m,\Phi_h^m) \in S_h \times S_h$ of (ψ^m,ϕ^m) following the steps below:

`Step 1:` Set

$$\Psi_h^0 := R_h^{\Psi}\psi_0 \quad \text{and} \quad \Phi_h^0 := R_h^{\Phi}\phi_0. \tag{II.1}$$

`Step 2:` Set

$$\Phi_h^1 := R_h^{\Phi}\big\{\phi_0 + k\,\phi_1 + \tfrac{k^2}{2}\left[\,\phi_0'' - \phi_0 - \lambda\,\phi_1 - \mathrm{Re}(\psi_0')\,\right]\big\}. \tag{II.2}$$

`Step 3:` Find $\Psi_h^1 \in S_h$ such that

$$\tfrac{\Psi_h^1-\Psi_h^0}{k} = i\,\mu\,\Delta_h\Big(\tfrac{\Psi_h^1+\Psi_h^0}{2}\Big) - \alpha\,\tfrac{\Psi_h^1+\Psi_h^0}{2} - i\,P_h\Big[\tfrac{\Phi_h^1+\Phi_h^0}{2}\;\tfrac{\Psi_h^1+\Psi_h^0}{2}\Big]. \tag{II.3}$$

`Step 4:` For $n = 1, \dots, N-1$, specify $(\Psi_h^{n+1}, \Phi_h^{n+1}) \in S_h \times S_h$ via the requirements

$$\tfrac{\Psi_h^{n+1}-\Psi_h^{n-1}}{2k} = i\,\mu\,\Delta_h\Big(\tfrac{\Psi_h^{n+1}+\Psi_h^{n-1}}{2}\Big) - \alpha\,\tfrac{\Psi_h^{n+1}+\Psi_h^{n-1}}{2} - i\,P_h\Big[\Phi_h^n\;\tfrac{\Psi_h^{n+1}+\Psi_h^{n-1}}{2}\Big] \tag{II.4}$$

and

$$\tfrac{\Phi_h^{n+1}-2\Phi_h^n+\Phi_h^{n-1}}{k^2} = \Delta_h\Big(\tfrac{\Phi_h^{n+1}+\Phi_h^{n-1}}{2}\Big) - \lambda\,\tfrac{\Phi_h^{n+1}-\Phi_h^{n-1}}{2k} - \tfrac{\Phi_h^{n+1}+\Phi_h^{n-1}}{2} - P_h\Big[\,\mathrm{Re}\,\big(\,(\Psi_h^n)'\,\big)\Big]. \tag{II.5}$$

We note that (II.4) and (II.5) are decoupled, and hence one can solve the corresponding linear systems in parallel.

B. Well-Posedness of the finite element method

In order to compute of the finite element approximations in `Step 3` and `Step 4` above, we have to solve numerically linear systems of algebraic equations, which are nonsingular according to the following lemma.

Lemma 2.1: The finite element approximations $\{\Psi^n\}_{n=1}^N$ and $\{\Phi_h^n\}_{n=2}^N$ are well-defined.

Proof. Let $\xi > 0$ and $\chi \in S_h$ be real valued. Then, we define linear discrete operators T_h^{Ψ}, $T_h^{\Phi} : S_h \to S_h$ by $T_h^{\Psi} v := v - i\,\xi\,\mu\,\Delta_h v + \alpha\,\xi\,v + i\,\xi\,P_h(\chi\,\psi)$ and $T_h^{\Phi} v := v - \frac{\xi^2}{2}\,\Delta_h v + \frac{\xi(\xi+\lambda)}{2}\,v$ for $v \in S_h$. The existence and uniqueness of $\{\Psi^n\}_{n=1}^N$ and $\{\Phi_h^n\}_{n=2}^N$ follows by showing that the operators T_h^{Φ} and T_h^{Ψ} have trivial null space. Let ψ^*, $\phi^* \in S_h$ such that $T_h^{\Psi}\psi^* = 0$ and $T_h^{\Phi}\phi^* = 0$. Then, we have $\mathrm{Re}(T_h^{\Psi}\psi^*, \psi^*)_0 = 0$ and $(T_h^{\Phi}\phi^*, \phi^*)_0 = 0$, which yield $\|\psi^*\|_0^2 + \alpha\,\xi\,\|\psi^*\|_0^2 = 0$ and $\|\phi^*\|_0^2 + \frac{\xi^2}{2}|\phi^*|_1^2 + \frac{\xi(\xi+\lambda)}{2}\,\|\phi^*\|_0^2 = 0$. Thus, we obtain $\psi^* = 0$ and $\phi^* = 0$, which ends the proof. □ ■

C. A priori bounds for the finite element approximations

For the finite element approximations $\{\Psi_h^m\}_{m=0}^N$ and $\{\Phi_h^m\}_{m=0}^N$, we can prove the following bounds (see [7])

$$\max_{0\le m\le N} \|\Psi_h^m\|_0 \le \|\Psi_h^0\|_0 \tag{II.6}$$

and

$$\max_{0\le m\le N} \{\,\|\Phi_h^m\|_1 + \|\Psi_h^m\|_1\,\} + \max_{0\le m\le N-1} \|\partial_k \Phi_h^m\|_0 \le C_1\,\Gamma_{kh}, \tag{II.7}$$

where

$$\Gamma_{kh} := \Big\|\tfrac{\Phi_h^1-\Phi_h^0}{k}\Big\|_0 + \|\Phi_h^1\|_1 + \|\Phi_h^0\|_1 + |\Psi_h^0|_1 + \|\Psi_h^0\|_0^3$$

Now, assuming enough regularity for the initial data we can bound Γ_{kh} as follows

$$\Gamma_{kh} \leq C_2 \left\{ |\psi_0|_1^3 + \|\phi_1\|_1 + \|\phi_0\|_3 + \|\psi_0\|_2 \right\}. \tag{II.8}$$

Here, C_1 and C_2 are real positive constants that depend on the data of the problem but they are independent of the initial conditions and the exact solution of the problem. We note that the a priori L^2 bound (II.6) is used in the proof of (II.7). Also, we note that the a priori bound (II.7)-(II.8) is important in the error analysis since allow us to handle the nonlinearity of the problem.

III. Convergence

A. Consistency

Let $\{\eta_\Psi^m\}_{m=0}^{N-1}$ and $\{\eta_\Phi^m\}_{m=1}^{N-1}$ be the consistency error functions defined by

$$\begin{aligned} \tfrac{\psi^1-\psi^0}{k} &= i\,\mu\,(\tfrac{\psi^1+\psi^0}{2})_{xx} - \alpha\,\tfrac{\psi^1+\psi^0}{2} - i\,\tfrac{\phi^1+\phi^0}{2}\,\tfrac{\psi^1+\psi^0}{2} - \eta_\Psi^0,\\ \tfrac{\psi^{n+1}-\psi^{n-1}}{2k} &= i\,\mu\,(\tfrac{\psi^{n+1}+\psi^{n-1}}{2})_{xx} - \alpha\,\tfrac{\psi^{n+1}+\psi^{n-1}}{2} - i\,\phi^n\,\tfrac{\psi^{n+1}+\psi^{n-1}}{2} - \eta_\Psi^n,\\ \tfrac{\phi^{n+1}-2\phi^n+\phi^{n-1}}{k^2} &= (\tfrac{\phi^{n+1}+\phi^{n-1}}{2})_{xx} - \lambda\,\tfrac{\phi^{n+1}-\phi^{n-1}}{2k} - \tfrac{\phi^{n+1}+\phi^{n-1}}{2} - \text{Re}\left[(\psi^n)_x\right] - \eta_\Phi^n \end{aligned} \tag{III.1}$$

for $n = 1, \ldots, N-1$. Assuming enough regularity for the exact solution of the problem and using Taylor's formula we can show that

$$\max_{0\leq m\leq N-1} \|\eta_\Psi^m\|_0 + \max_{1\leq m\leq N-1} \|\eta_\Phi^m\|_0 + \max_{2\leq m\leq N-2} \left\| \tfrac{\eta_\Psi^{m+1}-\eta_\Psi^{m-1}}{2k} \right\|_0 \leq C_3\,k^2, \tag{III.2}$$

where $C_3 > 0$ is a constant which is independent of k but it depends on the data and the exact solution of the problem.

B. Error estimates

Investigating the convergence of the numerical method described in Section II-A, we compare the finite element approximations Ψ_h^m and Φ_h^m with the elliptic projections $R_h^\Psi\psi^m$ and $R_h^\Phi\phi^m$, respectively. Thus, we derive the following H^1 superconvergence estimates (see [7]):

$$\|\Phi_h^1 - R_h^\Phi\phi^1\|_1 \leq C_4\,k^3, \tag{III.3}$$

$$\max_{0\leq m\leq N} \|\Psi_h^m - R_h^\Psi\psi^m\|_1 + \max_{0\leq m\leq N} \|\Phi_h^m - R_h^\Phi\phi^m\|_1 \leq C_5\,(k^2 + h^{r+1}), \tag{III.4}$$

where, the constants C_4 and C_5 depend on the data and the exact solution of the problem but they are independent of k and h. We note that the estimate (III.3) is used in the proof of (III.4) and it follows using Taylor's formula and (II.2). To obtain (III.4), first we formulate the corresponding error equations using (III.1), (II.3), (II.4) and (II.5), and then we build up a stability argument which is based on the often use of (II.7)-(II.8), (I.8), (I.9) and (III.2).

Finally, combining (III.4) and (I.8), we obtain an optimal order convergence in the L^2 and H^1 norms:

$$\max_{0\leq m\leq N} \left\{ \|\Phi_h^m - \phi^m\|_0 + \|\Psi_h^m - \psi^m\|_0 \right\} \leq C_6\,(k^2 + h^{r+1}),$$

$$\max_{0\leq m\leq N} \left\{ \|\Phi_h^m - \phi^m\|_1 + \|\Psi_h^m - \psi^m\|_1 \right\} \leq C_7\,(k^2 + h^r),$$

where C_6 and C_7 are constants independent of k and h.

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References

[1] S.C. Brenner and L.R. Scott: *The Mathematical Theory of Finite Element Methods*, Texts in Applied Mathematics 15, Springer-Verlag, New York, 1994.

[2] N. I. Karachalios, N. M. Stavrakakis and P. Xanthopoulos: *Parametric exponential energy decay for dissipative electron-ion plasma waves*, Z. Angew. Math. Phys. **56** (2005), pp. 218–238.

[3] V. Thomée: *Galerkin Finite Element Methods for Parabolic Problems*, Springer Series in Computational Mathematics 25, Springer-Verlag, Berlin Heidelberg, 1997.

[4] M.-F. Wheeler: *L_∞ estimates of optimal orders for Galerkin methods for one-dimensional second order parabolic and hyperbolic equations*, SIAM J. Numer. Anal. **10** (1973), pp. 908–913.

[5] P. Xanthopoulos: *Modelling and Asymptotic Behaviour of Dissipative Systems in Magnetic Fusion.*, Ph.D Thesis (in Greek), National Technical University, Athens, Greece, February 2003.

[6] V. E. Zakharov: *Collapse of Langmuir Waves,* Sov. Phys. JETP **35** (1972), pp. 908-912.

[7] G. E. Zouraris: *On the convergence of a linearly implicit finite element method for Klein-Gordon-Schrödinger-type system*, in preparation.