MULTIOPERATORS PRINCIPLE FOR CONSTRUCTING ARBITRARY-ORDER APPROXIMATIONS AND SCHEMES FOR PARALLEL CALCULATIONS

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Abstract

Multioperators strategy of constructing arbitrary-order approximations and schemes is presented. Its essence is forming linear combination of basis operators obtained by fixing sets of parameters in one-parameter families of compact approximations. In the case of hyperbolic systems and fluid dynamics equations, it is shown that the Compact Upwind Differencing (CUD) operators serve as a source of basis operators allowing to create highly accurate high resolution schemes. It is shown also how to obtain basis operators by modifying central compact approximations or by introducing inverse parameter-depending operators in standard formulas. Moreover novel one-parameter families of compact approximations with two- point inverse operators are considered as generators of basis operators. The resulting multioperators for typical grid functionals are outlined. Numerical examples are presented.

I. Introduction

A considerable part of modern computational problems require high accuracy numerical tools. In broad sence, high accuracy may be viewed as very accurate representations of grid functionals generated by linear operators (derivatives at specified nodes, definite integrals, interpolation/extrapolations operators etc.)using sets of nodes where functions are known. In the case of PDE's, high accuracy of their numerical solvers means very accurate representations of exact solution scales supported by the sets of nodes.

High accuracy computational schemes are especially important in the case of computational fluid dynamics and aeroacoustics problems. As examples, numerical simulations of turbulence, sound generation and propogation, various transport phenomena may be mentioned.

Quantitatively, errors of numerical analysis formulas may be measured first of all by approximation orders characterizing leading error terms in the corresponding Taylor expansion series. It may be viewed as a primary property of the formulas. In the case of derivatives discretizations, Fourier representation of the errors may be of importance. High orders mean that the Fourier components of the errors are very small in domains of long and possibly medium waves supported by meshes. Additionally, one may require that the largest wave numbers are represented accurately.

The standard way to increase approximation orders of numerical formulas approximating grid functionals in the case of structured meshes is to increase numbers of nodes in their stencils.

In the case of compact approximations, the well known drawbacks of excessively large stencils can be partly obviated by introducing inverses of operators with narrow stencils acting on other operators. For example, the Collatz and Numerov compact formulas provide fourth order accurate discretizations for the first and the second derivatives using only three-point operators. However considerable increase of approximation orders usually limited by the necessity of either enlarging stencils or adding complexity to the resulting algorithms.

An alternative approach to create arbitrary-order approximations was first presented by the author at the Manchester conference on parallel CFD in 1977 [6]. Its essence is forming linear combinations of basis operators from one-parameter families satisfying certain conditions by fixing distinct values of the parameter. The combinations were labelled as multioperators. In this way approximation orders can be increased simply by increasing the numbers of parameters M without changing the basis operators, the orders being linear functions in M.

When performing calculations of multioperators actions in the case of parallel machines, prescribed orders can be obtained by using processor whose numbers are equal to the numbers of parameters.

In the subsequent works [8],[9], [10], [4], the emphasize was placed on approximations to the first derivatives in convection,convection-diffusion and fluid dynamics types of equations. As basis operators, third- and fifth-order the so called Compact Upwind Differencing (CUD) one-parameter operators proposed previously by the author [7] were used by fixing distinct values of the parameter Later (for example, [10]), it was shown that basis operators can be constructed by forming compact approximations with artificially introduced parameters. As examples, central arbitrary-order multioperators for derivatives as well as other grid functionals were presented.

Though main emphasis in the multioperators investigations was placed on derivatives approximations, other grid functionals can be of interest when constructing high-order methods for PDE's or other applications. Examples are midpoint interpolations, integrals over cells etc.. To create multioperators for target functionals, it is sufficient to modify the corresponding standard formulas by introducing inverse operators depending on a parameter. The idea was used in [12].

Recently, novel families of one-parameter compact approximations to various functionals were suggested [14], the crucial point being the use of two-point inverse operators. They can generate basis operators for multioperators with some attractive properties.

Below we present the state-of-art of the methodology including the recent results. Special attention is given to basis operators defining main properties of the resulting multioperators. In Section 3, the emphasis is placed on CUD-based multioperators for fluid dynamics. It is also shown how to create other types of multioperators either by modifying central compact approximations to derivatives or by artificially introducing inverse operators depending on a parameter.

Sections 4 describes novel family of compact approximations with two-point inverse operators. The family possesses some attractive properties and can serve as a source of basis operators for multioperators approximating various grid functionals with desired orders. Illustrative numerical examples show high efficiency of the approach.

II. General formulation of the multioperators principle

Suppose that there is a family of operators $L_h(s)$ dependent on, at least, one parameter s and approximating an operator L on a grid ω_h. Suppose further that for a sufficiently smooth function $f \in U$ for each grid point with number j one has

$$[Lf]_j = L_h(s_i)[f]_j + \sum_{k=m}^{m+M-2} a_{kj} c_k(s_i) h^k + O(h^{m+M-1}) \tag{1}$$

where h is some parameter characterizing mesh size, a_{kj} are independent of h coefficients and $[\cdot]_h : U \to U_h(\omega_h)$ is a projection operator into a space U_h of grid functions defined on ω_h. The coefficients a_{kj} include high derivatives values at x_j.

We introduce a set of distinct values of s, ($s = s_i$, $i = 1, 2, \dots, M$) and a partition of unity γ_i, $i = 1, 2, \dots, M$. Considering Eq.(1) for each $s = s_i$, we multiply each resulting equality by γ_i and perform summation over i. Suppose now that the matrix A defined by

$$A = \begin{pmatrix} 1 & 1 & \dots & 1 \\ c_m(s_1) & c_m(s_2) & \dots & c_m(s_M) \\ c_{m+1}(s_1) & c_{m+1}(s_2) & \dots & c_{m+1}(s_M) \\ \vdots & \vdots & \ddots & \vdots \\ c_{m+M-2}(s_1) & c_{m+M-2}(s_2) & \dots & c_{m+M-2}(s_M) \end{pmatrix} \tag{2}$$

does not degenerate for all M. Then it is possible to find coefficients γ_i, $i = 1, 2, \dots, M$ such that (1) reduces to

$$[Lf]_j = \sum_{i=1}^{M} \gamma_i L_h(s_i)[f]_j + O(h^{m+M-1})$$

for arbitrary M. The linear combination $L_M = \sum_{i=1}^{M} \gamma_i L_h(s_i)$ is referred to as a multioperator while $L_h(s_i)$, $i = 1, 2, \dots, M$ may be viewed as basis operators.

The crucial point when constructing multioperators is the choice of one-parameter families for which the inequality $det A \neq 0$ is satisfied. Early multioperators where based on the third-order CUD families from [7] whose Taylor expansion series demonstrate clearly that the condition is met. The same result was obtained for the fifth-order CUD families.

At present, it is clear that any compact approximation with an inverse operator depending on a parameter has potential for being a generator of basis operators. To explain it, consider the simple case when $L_h(s)$ can be presented as $L_h(s) = (I + A_h(s))^{-1} B_h$ where I is the unity operator while the grid operators A_h and B_h are defined on an uniform mesh with the mesh size h as a summation over stensils with coefficients linearly depending on the parameter s (at least in the case of A_h). Considering actions of $L_h(s)$ on sufficient smooth functions, one can find that the coefficients for the powers of h^k in the relevant Taylor expansion series contain polynomials in s of degrees increasing with k. For example one can find that$c_m(s) = p_1(s)$, $c_{m+1}(s) = p_2(s), \dots$ in Eq. (1)where $p_k(s)$ are kth degree polynomial in s. It means that the system for γ coefficients can be reduced to a system for sums $\sum_{i=1}^{M} \gamma_i s_i$, $\sum_{i=1}^{M} \gamma_i s_i^2, \dots$. The last is known to be always solvable for distinct values of s_i. In the case of other possible structures of compact approximation operators, p_k can be polynomials in s^{-1} without affecting on the solvability. It is possible also that the system with matrix A is not reducible to that for the above sums and a solvability study is needed. Though it may result in some restrictions on the choice of s_i, simple parameters distributions inside an interval s_{min}, s_{max} were found to provide solvable systems and hence unique multioperators.

Concluding the Section, the important property of the multioperators is the possibility of calculating their actions in a parallel manner. Actually, the calculations can be carried out by calculating the actions of each basis operators by M processors simultaneously. The procedure is schematically shown in Fig.1. As seen, the execution time is independent of M if an idealized machine with negligible data exchange expenses is assumed.

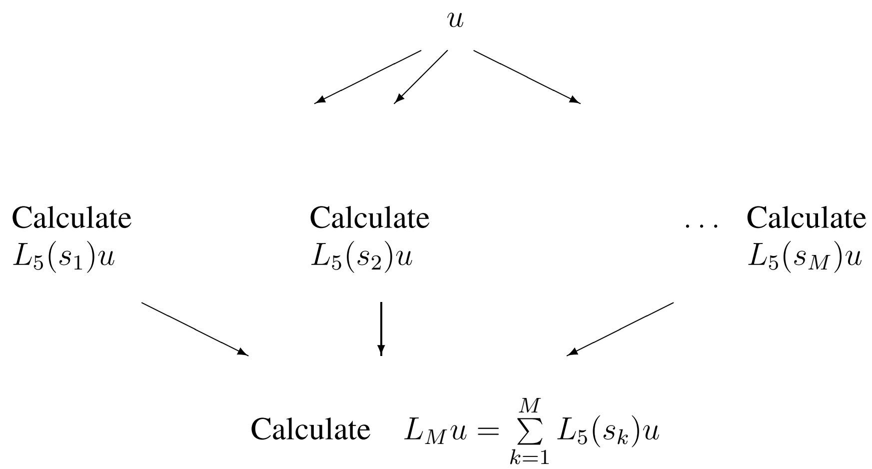

Fig. 1. Parallel algorithm

III. Multioperators with three-diagonal inverse operators

In the following, we suppose that $\omega_h = (x_j = jh,\ j = 0, \pm 1, \pm 2, \dots)$ is the uniform mesh with a mesh size h. Using the mesh, it is convenient to express any three-point operator as a linear combination of the unity operator I and three-point central differences Δ_0, Δ_2 defined by

$$\Delta_0 = T_1 - T_{-1}, \quad \Delta_2 = T_1 - 2I + T_{-1}, \quad T_{\pm 1} v_j = v_{j \pm 1}.$$

Now any compact approximation may be viewed now as a rational function of Δ_0 and Δ_2.

A. *Multioperators based on Compact Upwind Differencing operators AND RELATED SCHEMES*

Basis operators.

Historically, first multioperators were constructed to discretize convection terms of fluid dynamics equations. They were based on the third- and fifth- order Compact Upwind Differencing (CUD) operators from [7] approximating first derivatives which are functions of of Δ_0 and Δ_2 depending at least on one free parameter. They do not require grid functions values outside computational domains when applied to derivatives at internal nodes. In the condensed forms, the rth-order CUD families ($r = 3, 5$) look as

$$\begin{aligned} &\left[\frac{\partial u}{\partial x}\right]_j = L_m(s)u_j + O(h^m), \quad m = 3, 5, \\ &L_m(s) = (\Delta(s) + sR_1^{-1}(s)Q_1(s)\Delta_2)/h \quad \text{or} \\ &L_m(s) = R_2^{-1}(\Delta(s) + Q_2\Delta_2)/h \end{aligned} \tag{3}$$

where $\Delta(s) = 0.5(\Delta_0 - s\Delta_2)$, s is the upwinding parameter, h is a constant mesh size, $Q_1 = I$, $Q_2 = 0$ for $r = 3$, $Q_i = \tilde{Q}_i(I + \Delta_2/12)^{-1}$, $i = 1, 2$ for $r = 5$ while three-diagonal operators R_i, $\tilde{Q}_i$ can be expressed in terms of Δ_0 and Δ_2. The above two types of CUD may be viewed as additive and multiplicative corrections to the generic operator $\Delta(s)$.

To calculate the action, for example, of $L_5(s) = (\Delta(s) + sR_1^{-1}(s)Q_1\Delta_2)/h$ on a grid function, say on u_j, $j = 0, 1, \ldots, N$, it is sufficient to calculate first $v_j = (I + \Delta_2/12)^{-1}\Delta_2 u_j$ by inverting the tridiagonal matrix $(I + \Delta_2/12)$, then to calculate $w_j = R_1^{-1}(\tilde{Q}_1 v_j)$ by inverting again tridiagonal matrix and finally to calculate $\Delta(s)u_j + w_j$. As seen, three-point operators only are involved in the calculations thus using only internal grid values and requiring $O(N)$ operations. It is supposed that boundary conditions are formulated to perform the inversions.

Let U_h be the Hilbert space of grid functions $u_h = (u_j,\ j = 0, \pm 1, \pm 2, \ldots)$ with summable squares. Using the inner product defined by $(u_h, v_h) = h\sum_{j=-\infty}^{\infty} u_j v_j$, it can be shown that upon presenting $L_m(s)$ as sums of the skew-symmetric $L_m^{(1)}$ and self-adjoint $L_m^{(0)}$ parts ($L_m = L_m^{(1)} + L_m^{(0)}$), one can write

$$L_m^{(1)}(s) = L_m^{(1)}(-s), \quad L_m^{(0)}(s) = -L_m^{(0)}(-s) \tag{4}$$

In terms of the conjugate operation, Eqs.(4) can be written as $L_m(s)^* = -L_m(-s)$. It was shown in [7] that L_m is a positive operator, that is $(L_m u_h, u_h) > 0$ (which means that $(L_m^{(0)} u_h, u_h) > 0$), $u_h \in U_h$ if $s > s_* > 0$ where $s_* = 0$ for the third-order CUD and the multiplicative fifth-order correction and s_* is about unity for the fifth-order CUD. Considering the corresponding Fourier space, the real part of the Fourier transform $\hat{L}_m(s)$ of $L_m(s)$ can be shown to be a non-negative and non-positive function of the Fourier variable for $s > s_*$. It satisfies $\mathrm{Re}\hat{L}_m(s) = -\mathrm{Re}\hat{L}_m(-s)$, the imaginary part being invariant under the transformation $s \to -s$.

Another important property of the CUD operators is the possibility of presenting their actions in the conservative form. For example,

$$L_5(s)u_j = (q_{j+1/2} - q_{j-1/2})/h \tag{5}$$

where $q_{j+1/2}, j = 0, 1, \ldots, N-1$ can be calculated in the above described manner with changing $\Delta_0 u_j$ and $\Delta_2 u_j$ by $(u_{j+1} + u_j) - (u_j + u_{j-1})$ and $(u_{j+1} - u_j) - (u_j - u_{j-1})$ respectively. Thus

$$q_{j+1/2} = G(s)u_j = \frac{u_{j+1} + u_j}{2} - s\frac{u_{j+1} - u_j}{2} + sR_1^{-1}(s)Q_1(s)\frac{u_{j+1} - u_j}{2}. \tag{6}$$

Thus two different fluxes $q_{j+1/2}^{+}$ and $q_{j+1/2}^{-}$ can be calculated for $s > 0$ and $s < 0$ respectively.

Multioperators

Considering the Taylor expansion series for the CUD operators actions, one can find that they look like Eq. (1) with the coefficients $c_k(s)$ given by either $c_k(s) = p_{k-m+1}(s)$, $k = m, m+1, m+2, \ldots$ or $c_k(s) = p_{k-m+1}(s^{-1})$, $k = m, m+1, m+2, \ldots$ where $p_n(x)$ is a nth degree polynomial in x.

Setting $s = s_1, s_2, \ldots, M$, one obtains a linear system for the coefficients γ_i defining the corresponding multioperator. Its matrix A can be easily reduced to the Vandermonde type of matrix whose rows are powers or inverse powers od s. It guarantees the existence and uniqueness of the $M+m-1$-order multioperators approximation to first derivatives and provide analytical solutions for the coefficients. To improve the conditioning of the resulting system (the matrices are ill-conditioned for large M), we define them as zeroes of the Chebyshev fifth-order polynomial for the interval $[s_{min}, s_{max}]$: $0 < s_{min} \leq s_1 < \cdots < s_M \leq s_{max}$. Now the multioperators depend on two parameters, s_{min}, s_{min}, thus their investigation being greatly simplified. The only exception is the CUD option from (3) used in [13]. The expansion for that option looks as

$$\begin{array}{l} L_5(s_i)[u]_j = [\partial u/\partial x]_j + \sum_{k=5}^{l} p_{k-4}(s_i)[\partial^{k+1}u/\partial x^{k+1}]_j h^k + O(h^{l+1}), \\ p_{k-4}(s_i) = a_k s_i + \sum_{l=1}^{k-4} b_{kl} s_i^{-l}, \quad i = 1, 2, \ldots, M \end{array} \tag{7}$$

and the corresponding system is

$$\sum_{i=1}^{M} \gamma_i = 1, \sum_{i=1}^{M} \gamma_i p_1(s_i) = 0, \sum_{i=1}^{M} \gamma_i p_2(s_i) = 0, \ldots \sum_{i=1}^{M} \gamma_i p_{M-1}(s_i) = 0$$

It was found to be solvable for fixed distributions inside the interval $[s_{min}, s_{max}]$.The details can be found in [13].

Denoting the L_m-based multioperator by $L_{m,M} = \sum_{i=1}^{M} \gamma_i L_m(s_i)$, the following equality similar to that for the basis operators can be proved

$$\begin{array}{l} L^{(1)}_{m,M}(s_1, s_2, \ldots, s_M) = L^{(1)}_{m,M}(-s_1, -s_2, \ldots, -s_M) \\ L^{(0)}_{m,M}(s_1, s_2, \ldots, s_M) = -L^{(0)}_{M}(-s_1, -s_2, \ldots, -s_M) \end{array} \tag{8}$$

or $L_{m,M}(-s_1, -s_2, \ldots, -s_M) = L^{*}_{m,M}(s_1, s_2, \ldots, s_M$. The above property follows from that of the basis operators. However the positivity of the last does not necessary means the positivity of the multioperators. To preserve the property for assumed distributions of parameters, one should find s_{min}, s_{max} for which

$$L_{m,M}(s_{min}, s_{max}) > 0.$$

The problem can be solved by calculating the real part of the Fourier transform $\hat{L}_{m,M}(\alpha)$ of $L_{m,M}$ where $\alpha = kh$ and k is the wave number. The calculations for each grid point of a grid in the (s_{min}, s_{max}) plane allows one to check if they are positive or negative for $\alpha \in [0, \pi]$. Supposing that the solution exists, any resulting "good" pair s_{min}, s_{max} can serve as the multioperators parameters providing positive or negative approximations. In the latter case it is sufficient to use $-s_{min}, -s_{max}$ to guarantee multioperators positivity Domains of positivity for the multioperators based on the third- and fifth-order CUD are described in [9],[4],[10] and [13].

Multioperators CUD-based schemes for hyperbolic conservation laws.

Using the multioperators pairs defined by admissible values s_{min}, s_{max} and $-s_{min}, -s_{max}$ for convection, convection-diffusion equations, hyperbolic and fluid dynamics systems allows one to construct robust stable schemes with positive operators (in the frozen coefficients sense). Optionally, either finite difference or finite-volume type approximations can be constructed in non-linear cases. One of the possible options [7], [11] looks as follows. Consider systems of conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{f}(\mathbf{u})}{\partial x} = 0 \tag{9}$$

where $\mathbf{u}$ and $\mathbf{f}$ are p-components vectors. Suppose that a set of parameters defined by s_{min}, s_{max} is found to give $L^{+} = L_{m,M}(s_{min}, s_{max}) > 0$. Denoting $L^{-} = L_{m,M}(-s_{min}, -s_{max})$, the semi-discretized scheme for Eq. (9) in the index-free form can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \mathcal{L}u = 0, \quad \mathcal{L}u = \frac{1}{2}\Big(L^{+}(\mathbf{f}(\mathbf{u}) + C\mathbf{u}\Big) + L^{-}\Big(\mathbf{f}(\mathbf{u}) - C\mathbf{u}\Big)\Big) = 0 \tag{10}$$

where $C = const > 0$. Now the spatial discretization of (9) can be presented as

$$\mathcal{L} = (1/2)(L^{+} + L^{-})\mathbf{f}(\mathbf{u}) + (1/2)C(L^{+} - L^{-})\mathbf{u}$$

The sum and the difference of the operators can be readily recognized as the skew-symmetric and the self-adjoint parts of our multioperators respectively, the latter being a high-order dissipative mechanism playing role of a built-in filter of spurious oscillations. The conservative property of the scheme follows from Eq. (6) and the action of $\mathcal{L}$ can be presented as a difference of numerical fluxes across cells boundaries.

Assuming $\mathbf{f}(\mathbf{u}) = A\mathbf{u}$ where A is a constant symmetric matrix with real eigenvalues, it is easy to see that (10) is conditionally stable in the L_2 norm generated by the introduced inner product. Specifying time stepping procedures, one can construct various conditionally or unconditionally stable fully discretized schemes. In particular, the Runge-Kutta technique is appropriate for unsteady problems while the following two-level scheme can be used to get steady state solutions.

$$(I + \tau L_1)\frac{u^{m+1} - u^m}{\tau} + \mathcal{L}u^m = 0, \quad t_m = m\tau,\, m = 0, 1, 2, \ldots \tag{11}$$

where L_1 is a preconditioner admitting relatively simple inversion of the time stepping operator and preserving stability of the scheme. One of the possible options is to choose L_1 as a first-order approximation to the x-derivative.

Considering (9) as the scalar equation ($p = 1$) with $f(u) = au$, $a = const > 0$, one can estimate in the standard way dispersion and dissipation properties of the semi-discretized scheme with $\mathcal{L}$ operator. To perform the analysis, we use, for example, the space $\bar{U}_h$ of bounded grid functions. Now operator $\mathcal{L} : \bar{U}_h \to \bar{U}_h$ has the eigenfunctions $w_n = \exp(i\alpha n)$, $0 \le \alpha \le 2\pi$, $n = 0, \pm1, \pm2, \ldots$, $w_n \in \bar{U}_h$. Assuming w_n to be the initial value for (9), the solution of (10) can be readily obtained to give

$$u_n(t) = \mathrm{e}^{-Cad(\alpha)t/h}\mathrm{e}^{i(\alpha n - a\varphi(\alpha)t/h)} = \mathrm{e}^{-Cad(\alpha)t/h}\mathrm{e}^{ik(x_n - a\varphi(\alpha)t/\alpha)} \tag{12}$$

where C is the flux splitting constant, $d(\alpha)/h > 0$ and $\varphi(\alpha)/h$ are the real and imaginary parts of the eigenvalues while $k = \alpha/h$ is the wave number and $x_n = nh$. The term $a\varphi(\alpha)/\alpha = ar(\alpha)$ may be viewed as the numerical phase velocity and the deviation of $r(\alpha)$ from unity defines the phase errors introduced by the scheme. The positive function Cd characterizes the attenuation of the initial harmonics during time interval $t = h/a$ and may be considered as a measure of the amplitude errors.

In the case of a Kth-order multioperator, the phase and amplitude errors can be estimated by $r(\alpha)$, $d(\alpha) = O(\alpha^{K+1})$ when $\alpha \to 0$. Using s_{min}, s_{max} as controlling parameters,it is possible to have the errors which are small up to large values of α.

One can easily extend schemes (10) and (11) to the case of the Euler equations written as conservation laws in curvilinear coordinates. In that case, it is sufficient to construct $\mathcal{L}$ operators corresponding to each spatial coordinate. As to the Navier-Stokes equations, the terms with viscosity coefficients can be approximated quite independently with desired order. In particular, centered compact approximations can be used.

Example: ninth-order multioperators scheme

Details of the CUD-based multioperators schemes can be found in [9], [4], [10]. We present here the recent results from [13] concerning the ninth-order multioperator with the $L_5(s_i)$ basis operators ($i = 1, 2, \ldots, 5$) given by Eq. (7).

Fixing an uniformly distributed five parameters $s_{min} = s_1 < s_2 <, \ldots < s_5 = s_{max}$, it was found that the system for the γ coefficients is solvable and the multioperator defined by

$$L_{59}(s_1, s_5) = \sum_{k=1}^{5} \gamma_k L_5(s_k)$$

do exist. It was found also that the real part of the Fourier transform $d_9(\alpha; s_{min}, s_{max})h = \hat{L}_{59}(s_{min}, s_{max})$ is negative if the pair (s_{min}, s_{max}) belongs to some domain in the parameters plane. In that domain, the s_{min} and s_{max} values can be used to control the dispersion and dissipation properties in the case of advection equations with constant coefficients.

The functions $r(\alpha) = r_m(\alpha)$, $m = 7, 9$, $\alpha = kh$ are shown in Fig.3 for $s_1 = .82, s(3) = 10$ and $s_1 = .835, s(5) = 15$ in the case of the seventh-order and ninth-order multioperators respectively [13].

Though the curves for $M = 3$ and $M = 5$ look almost identical, the numerical values of the phase errors were found to be considerably smaller in the latter case for the long and medium waves, the estimates being $O(\alpha^8)$ and $O(\alpha^{10})$ for $M = 3$ and $M = 5$ respectively.

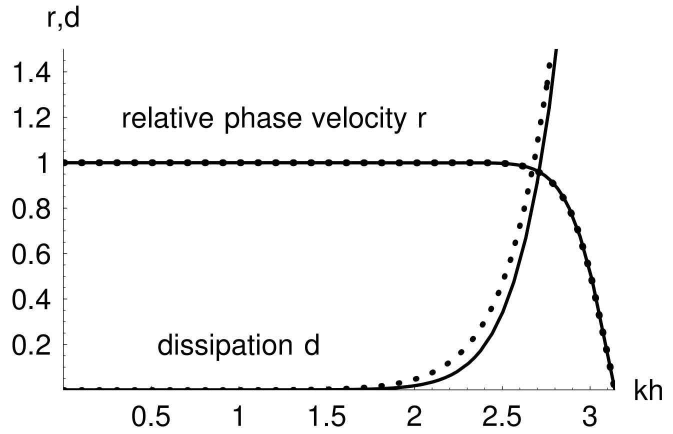

Fig. 2. Relative phase velocity and dissipation parameter vs. dimensionless wave number.Solid and dashed lines correspond to ninth- and seventh-order multioperators

As seen, the phase and amplitude errors are small for quite large domain of wave numbers supported by meshes. This property is very important in the case of areoacoustics applications.

As an illustrative example, consider the periodic IVP problem for the Burgers equation which is often used as a testing one (see, for example [1])

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\frac{u^2}{2} = 0, \quad -1 \leq x \leq 1 \tag{13}$$

$$u(0, x) = 1 + 0.5\,\sin(\pi x), \quad -1 \leq x \leq 1$$

The exact solution up to $t = 2/\pi$ is smooth. It can be obtained using an iterative procedure described in [1]) with the machine precision. Its values at grid points will be denoted by u_j^{ref}.

The calculations with the splitting constant $C = 1$ were carried out using L_{59} operator and the fourth-order Runge-Kutta method. The accuracy was estimated using the discrete maximum norm giving numerical solution errors and approximate mesh convergence orders as

$$E_c(n) = \max_j |u_j - u_j^{ref}|, \quad k_c = \log_2 \frac{E_c(n)}{E_c(2n)}.$$

Table 2 [13] gives a general impression concerning accuracy of the scheme as compared with that of other multioperators CUD-based methods. In the Table, the solution errors at $t = 0.3$ are presented for several meshes with the number of nodes n. The methods are the seventh-order scheme from [13] (L_{57}) and the previous versions of ninth-order multioperators schemes (denoted here by L_{59a} and L_{39}). The last are based on another fifth-order operator from [7], [11] (denoted here by L_{5a}) and one of the third-order CUD basis operators from [7].

For comparison, the data from [5] obtained with the previous versions of ninth-order multioperators (denoted here by L_{59a} and L_{39}) are included in the Table. The multioperators are based on another fifth-order operator from [7], [11] (denoted here by L_{5a}) and one of the third-order CUD basis operators from [7]. The Table contains also the results of calculations with the fifth-order WENO scheme presented in [5]. As seen, the considered scheme with L_{59} outperforms all other schemes demonstrating remarkably high accuracy. Note that the claimed convergence orders of the schemes settle earlier when doubling the

TABLE I

SOLUTION ERRORS FOR MULTIOPERATORS CUD-BASED SCHEMES

n		8	16	32	64	128	256
WENO-5	E_c	6.47e-2	1.25e-2	1.20e-3	9.50e-5	3.31e-6	8.66e-8
	k_c		2.4	3.4	3.7	4.8	5.3
L_{5a}	E_c	3.99e-2	6.10e-3	4.35e-4	1.63e-5	5.11e-7	1.56e-8
	k_c		2.71	3.81	4.74	5.00	5.03
L_{39}	E_c	4.96e-2	7.86e-3	3.85e-4	9.01e-6	7.76e-8	3.58e-10
	k_c		2.66	4.35	5.42	6.86	7.76
L_{59a}	E_c	3.30e-2	2.89e-3	1.30e-4	2.17e-6	1.30e-8	3.46e-11
	k_c		3.51	4.47	5.91	7.38	8.55
L_{57}	E_c	1.82e-2	2.14e-3	3.61e-5	2.21e-7	8.03e-10	3.38e-12
	k_c		3.1	5.9	7.3	8.1	7.9
L_{59}	E_c	1.74e-2	1.86e-3	2.16e-5	5.02e-8	4.7e-11	9.96e-14
	k_c		3.2	6.4	8.7	10	9.2

number of nodes as compared with those for other ninth-order multioperators. The last were found to provide the ninth-order mesh-convergence only in the case of more refined meshes than those presented in the Table 1.

Though Table 1 gives a general idea of the multioperators performances, it is worth noting that the presented results concern particular choices of the involved parameters which are not necessarily optimal for each scheme . Thus the possibility exists of further decreasing the corresponding solution errors. In turn, it may influence the estimates of the relative efficiency of the schemes.

B. Multioperators based on artificially created one-parameter approximations

The above described multioperators exploit natural families of compact approximations depending on the upwinding parameter s and aimed at discretization of first derivatives in hyperbolic conservation laws or fluid dynamics equations. However the multioperators principle can be applied to more general cases of grid functionals for which one-parameter compact approximations are not readily available. To do so, one can either include a parameter into inverse operators in existing parameter-free compact formulas or multiply non-compact ones by a parameter-dependant inverse operators.

Multioperators with central compact differencing operators.

In the context of PDE's solvers, it makes sense in many cases to use central approximations to derivatives. Examples are schemes for slow flows, discretizations of diffusion terms and the Poisson operator in the vorticity-stream function formulation, schemes for elasticity equations etc.

Families of central high-order compact approximations to various grid functionals (and, in particular, to derivatives at grid points) were proposed by Lele in his well-known paper [3]. In the simplest case of the first or second derivatives and three-point stencils, the differencing formulas in our notations look as

$$\frac{1}{2h}(I+\frac{1}{6}\Delta_2)^{-1}\Delta_0[f]_j = [\frac{\partial f}{\partial x}]_j + O(h^4),$$
$$\frac{1}{h^2}(I+\frac{1}{12}\Delta_2)^{-1}\Delta_2[f]_j = [\frac{\partial^2 f}{\partial x^2}]_j + O(h^4).$$

They are known as Collatz and Numerov formulas respectively (note that the Collatz operator can be obtained by setting $s = 0$ in the operator in the left hand side of Eq.(A1)). Due to the lack of free parameters, it is not possible to use them as basis operators. However, it is possible to obtain the necessary basis operators by inserting a parameter in the inverse operators. The corrected formulas then define one-parameter families of the second-order compact approximations. Denoting the parameter by c, the corresponding $D_1(c)$ and $D_2(c)$ operators and the expansions for their actions read

$$\begin{aligned} D_1(c)[f]_j &= \frac{1}{2h}(I+c\Delta_2)^{-1}\Delta_0[f]_j \\ &= [\frac{\partial f}{\partial x}]_j + (\frac{1}{6}-c)h^2 f_j^{(3)} + (\frac{1}{120}-\frac{c}{4}+c^2)h^4 f_j^{(5)} + O(h^6), \\ D_2(c)[f]_j &= \frac{1}{h^2}(I+c\Delta_2)^{-1}\Delta_2[f]_j \\ &= [\frac{\partial^2 f}{\partial x^2}]_j + (\frac{1}{12}-c)h^2 f_j^{(4)} + (\frac{1}{360}-\frac{c}{6}+c^2)h^4 f_j^{(6)} + O(h^6), \end{aligned} \qquad (A3)$$

TABLE II

SOLUTION ERRORS FOR CENTRAL MULTIOPERATORS SCHEMES IN THE CASES OF THE POISSON AND BIHARMONIC EQUATIONS

$N \times N$	3×3	5×5	11×11	23×23
Poisson eq.	2.31D-6	3.28D-7	6.23D-9	1.01D-10
Biharmonic eq.	4.62D-6	6.57D-7	1.24D-8	2.01D-10

the coefficients for further terms being polynomials of successively increasing degrees.

Fixing M parameters $c_1, c_2, \ldots, c_M$ and retaining more terms in the expansions, $2M$th-order multioperators with $\gamma_i^{(1)}$ and $\gamma_i^{(2)}$ coefficients $D_{1M} \sum_{i=1}^{M} \gamma_i^{(1)} D_1(c_i)$ and $D_{2M} \sum_{i=1}^{M} \gamma_i^{(2)} D_1(c_i)$ can be constructed as compared with $(M+m-1)$th-order ones in the case of mth-order CUD basis operators ($m = 3, 5$). Thus the central multioperators for the same M are superior over non-central ones from the viewpoint of their orders and accuracy.

In the CFD context, schemes with central multioperators for the first derivatives can be efficient when supplied with some artificial high-order dissipative mechanisms. More details and numerical examples showing highest accuracy of schemes withy central multioperators can be found in [12].

As an illustration, consider the following BVP problem for the Poisson equation

$$\Delta u = -2\pi^2 \sin \pi x \sin \pi y,\; x, y \in \Omega = [0,1] \times [0,1],\; u|_{\partial\Omega} = 0 \tag{14}$$

Its exact solution is $\sin \pi x \sin \pi y$. The same exact solution can be obtained for the biharmonic equation describing bending of a square plate

$$\Delta\Delta u = 4\pi^4 \sin \pi x \sin \pi y,\; x, y \in \Omega \tag{15}$$

with boundary conditions $u|_{\partial\Omega} = 0, \partial^2 u/\partial x^2 = 0$ for $x = 0, 1,\; \partial^2 u/\partial y^2 = 0$ for $y = 0, 1$

Sixth-order operators D_{2M}^x and D_{2m}^y ($M = 3$) for x and y coordinates were used to approximate the Laplace operator with some restrictions on parameters c_1, c_2, c_3 guaranteeing negativity of the multioperators [12]. In the case of problem (15), the biharmonic operator was considered as the square of the Laplace one.

The direct FFT solver was used to obtain the numerical solutions. The L_2 norms of the solutions errors for both problem and several uniform meshes are shown in Table 2 ($N \times N$ stands for the number of internal grid points).

As seen from the Table 2, the numerical solution accuracy is very high even if only 3 grid points are placed in each spatial directions. The mesh-convergence orders when increasing the numbers of grid points N from 3 to5 etc. were found to be approximately $4.8,\; 5.7.\; 5.9$ for both problems.

Other types of multioperators.

Though main emphasis in the multioperators investigations was placed on derivatives approximations, other grid functionals can be of interest when constructing high-order methods for PDE's or other applications. Examples are midpoint interpolations, integrals over cells etc.. To create multioperators for target functionals, one can modify existing compact parameter-free operators in the above described manner. For example, the operators described in [3] can be used for this purpose.

Having in mind quite general case of approximate formulas for grid functional $[Lu]_j$, one can start from some "conventional" at least first-order approximation

$$[Lu]_j \sim \bar{L}_h u_j = \sum_k a_k u_{j+k}$$

where the summation performed over some stencil with some a_k coefficients. To create one-parameter formulas, one can use instead of $\bar{L}_h$ a modified operator

$$L_h = (I + p\Delta_0 + q\Delta_2)^{-1} \bar{L}_h \tag{16}$$

where either p or q coefficient is a free parameter while the rest of the involved coefficients are supposed to be chosen to maximize the approximation order of the formula. It can be accomplished by using expansion series for the truncation error $[Lu]_j - L_h[u]_j$.

A non-central approximation with a free parameter s corresponds to the choice $p = s$ defining other coefficients maximizing the order. Eq. (16) may be viewed as a multiplicative correction for $\bar{L}_h$. Another option ia an additive correction of the form

$$L_h = \bar{L}_h + (I + p\Delta_0 + q\Delta_2)^{-1}\bar{B}_h \tag{17}$$

where B_h is given by $B_h u_j = \sum_k b_k u_{j+k}$ with the coefficients b_k to be determined via the previous procedure. As to the choice of $\bar{L}_h$, either a prescribed well known formula can be used or the coefficients a_k can be found in the process of annihilation of low order truncation errors terms.

In the case of additive corrections, one has more degrees of freedom to maximize the order of L_h.

The third-order CUD operators have the forms given by Eqs. (16), (17) with $\bar{L}_h = \Delta(s)$. The fifth-order CUD are also corrections for $\Delta(s)$ with the corrections terms containing inverse operators.

As a simple example, consider midpoint functional $[Lu]_j = u(x_{j+1/2})$. Its approximations are often used in various schemes for PDE's. Following the above described strategy, multiplicative one-parameter compact approximation to midpoint values can be obtained as

$$[Lu]_j = (I + (\frac{1}{4} + s)\Delta_0 + (\frac{1}{4} + \frac{s}{2})\Delta_2)^{-1}(I + s\Delta_0 + (\frac{1}{8} - \frac{s}{2})\Delta_2)[u]_j + O(h^4).$$

Once one-parameter operators are defined, multioperators for fixed sets of the parameter values can be constructed in the straightforward way.

IV. Multioperators with two-diagonal inverse operators

A. *Basis operators*

Recently, novel families of one-parameter compact approximations to various functionals were suggested [14], the crucial point being the use of two-point inverse operators. They can generate basis operators for multioperators with some attractive properties.

As previously, we consider $[Lu]_j$ as the action of a linear operator L associated with a node j on a function $u(x)$ of the continuous argument x. For example, $[Lu]_j$ can be viewed as integrals with respect to x between the limits in a vicinity of x_j, as values for shifted arguments $u(x_j + kh)$ with a fixed parameter k, as derivatives at x_j etc..

Following the strategy of using three-point stencils only, suppose that the functional is approximated by three-point formula which general form can be written as $\bar{L}_h u_j = (aI + d\Delta_0 + e\Delta_2)u_j$ whith a, d, e coefficients. Consider now the grid operators of the form

$$N_l(c) = (I + c(I - T^{-1}))^{-1} \quad N_r(c) = (I + c(I - T^{1}))^{-1},$$

where c is a free parameter.

The actions $w_j = N_k u_j, k = l, r$ of N_l and N_r on known grid functions u_j in the case of bounded domains $\Omega = \{x_0 \leq x \leq x_n\}$ can be calculated using the following procedures.

$$\begin{aligned} &w_j = \alpha w_{j-1} + u_j/(1+c), \ w_0 \text{ is given}, \quad j = 1, 2, \dots n \\ &w_j = \alpha w_{j+1} + u_j/(1+c), \ w_n \text{ is given}, \ j = n, n-1, \dots, 0 \\ &\alpha = c/(1+c), \end{aligned} \tag{18}$$

Supposing that $|c| << 1$, the impact of the initial values u_0 and u_n on the values u_k and u_{n-k} decays very rapidly (as $O(c^k)$)) with increasing the distances kh from the boundaries.

Using the skew-symmetric and self-adjoint operators Δ_0 and Δ_2, one can cast the inverses of N_l and N_r in the form

$$N_l^{-1} = I + c(\Delta_0 - \Delta_2)/2, \quad N_r^{-1} = I - c(\Delta_0 + \Delta_2)/2$$

.

Using upper indexes "(0)" and "(1)" for self-adjoint and skew-symmetric components of operators respectively, one can see that

$$(N_l^{-1})^{(0)} = (N_r^{-1})^{(0)} = I - c\Delta_2 > 0$$

for $c > -.5$ and

$$(N_l^{-1})^{(1)} = -(N_r^{-1})^{(1)} = c\Delta_0,$$

the positivity of the self-adjoint components being due to the operators inequality $-\Delta_2 \leq 4I$. Thus $N_r^{-1} = (N_l^{-1})^*$. Using the equality, it can be proved that for $c > -0.5$ the following is true.

$$N_l^{(0)} = N_r^{(0)} > 0, \; N_l^{(1)} = -N_r^{(1)}$$

and therefore $N_r = N_l^*$.

Following the CUD architecture and the previous representation Eqs.(16), (17), we construct now additive and multiplicative corrections to $\bar{L}_h$ using N_l and N_r.

$$L_l(c) = \bar{L}_h + N_l(c)(a_l\Delta_0 + b_l\Delta_2), \quad L_r(c) = \bar{L}_h + N_r(c)(a_r\Delta_0 + b_r\Delta_2), \tag{19}$$

or

$$\tilde{L}_l(c) = N_l(c)(\bar{L}_h + \tilde{a}_l\Delta_0 + \tilde{b}_l\Delta_2), \quad \tilde{L}_r(c) = N_r(c)(\bar{L}_h + \tilde{a}_r\Delta_0 + \tilde{b}_r\Delta_2), \tag{20}$$

where a_l, b_l, a_r, b_r , $\tilde{a}_l$, $\tilde{b}_l$, $\tilde{a}_r$, $\tilde{b}_r$ are parameters generally depending on c. The parameters are supposed to be obtained by maximizing the approximation orders of $L_l(c), L_r(c)$. It can be accomplished by setting to zero as many terms in the Taylor expansion series for the operators actions on sufficiently smooth functions as possible. Supposing that the parameters with the "l" indexes are found, their counterparts with the "r" indexes can be determined using the following considerations based on the conjugate properties of the involved operators.

Let $L_{h,r} = L_{h,l}^*$. Then $L_l^* = \bar{L}_{h,r} + N_r(-a_l\Delta_0 + b_l\Delta_2)$ and it is natural to set $a_r = -a_l$ and $b_r = b_l$ thus obtaining $L_r = L_l^*$. If $\bar{L}_{h,r} = -L_{h,l}^*$ then $L_r = -L_l^*$ with $a_r = a_l$ and $b_r = -b_l$. Thus in contrast to the CUD operators, the operators pairs are formed by left and right operators rather than by changing signs of the parameters.

The one-parameter operators families defined by (19), (20) should not be confused with compact approximations requiring two-diagonal inversions proposed previously. For example, the third-order approximation to first derivatives (CUD-3) defined and investigated in depth in [7] looks as

$$(I + \frac{1}{6}\Delta_0 - \frac{s}{4}\Delta_2)^{-1}\Delta(s))/h,$$

its early version with $|s| = 1$ being dated back to 1973 [7]). The three-diagonal inverse operator can be reduced to two-diagonal one simply by setting $|s| = 2/3$ thus removing the free parameter. In the particular case of the derivatives discretizations, it corresponds to multiplicative corrections in Eqs. (19), (20) with $c = 1/3$.

The merits of the above approximations is due to their ability to provide "one-sided" approximations which for $c << 1$ do not practically "feel" boundary conditions needed to perform calculations defined in Eq.(18). Another advantage over compact formulas which need inversions of three or more points operators is very small number of arithmetic operations per node.

Midpoint iterpolation. We return now to the the compact approximations to midpoint values considered in the previous Section in the context of three-point inverse operators. As a starting operator $\bar{L}_h$, the "left" and "right" approximations

$$u(x_{j+1/2}) \approx (3u(x_j) - u(x_{j-1}))/2, \quad u(x_{j-1/2}) \approx (3u(x_j) - u(x_{j+1}))/2.$$

can be used. Adding the correction terms with N_l and N_r operators and the coefficients a_l, b_l, a_r, b_r chosen to get the highest approximation orders, one can arrive at the "left" and "right" midpoint operators L_h denoted here by M_l and M_r respectively.

$$\begin{aligned} M_l &= (3I/2 - T_1/2 + (3/8 - 1/16c)\Delta_2 + N_l\Delta_2/16c, \\ M_r &= (3I/2 - T_{-1}/2 + (3/8 - 1/16c)\Delta_2 + N_r\Delta_2/16c. \end{aligned} \tag{21}$$

It can be shown that they are forth-order accurate, the first three terms of the Taylor expansion series being

$$\begin{aligned} M_l[u]_j &= u(x_{j+1/2}) + (\tfrac{5+8c}{128}h^4u_x^{(4)} - \tfrac{3+16c+16c^2}{256}h^5u_x^{(5)})|_{x=x_j} + O(h^6), \\ M_r[u]_j &= u(x_{j-1/2}) + (\tfrac{5+8c}{128}h^4u_x^{(4)} + \tfrac{3+16c+16c^2}{256}h^5u_x^{(5)})|_{x=x_j} + O(h^6). \end{aligned} \tag{22}$$

Taking into account the equalities $N_r = N_l^*$ and $T_1 = T_{-1}^*$, one can deduce from (21) that $M_r = M_l^*$. It explains why the terms with the fifth-order derivatives in Eqs. (22) have opposite signs.

Approximations to shift operators. Extrapolation formulas are often needed to create boundary closures for difference schemes.

Consider simple extrapolation formulas

$$u(x_{j+2}) \approx 2u(x_{j+1}) - u(x_j), \quad u(x_{j-2}) \approx 2u(x_{j-1}) - u(x_j).$$

Performing additive corrections, one can get the following forth-order compact approximations denoted by E_l and E_r.

$$E_l = 2T_1 - I + (1 + 1/c)\Delta_2 - N_l\Delta_2/c \quad E_r = 2T_{-1} - I + (1 + 1/c)\Delta_2 - N_r\Delta_2/c.$$

The operators are related through $E_r = E_l^*$ and the corresponding Taylor expansion series read

$$\begin{aligned} E_l[u]_j &= u(x_{j+2}) - ((1+c)h^4u_x^{(4)} + (c+c^2)h^5u_x^{(5)})|_{x=x_j} + O(h^6), \\ E_r[u]_j &= u(x_{j-2}) - ((1+c)h^4u_x^{(4)} - (c+c^2)h^5u_x^{(5)})|_{x=x_j} + O(h^6). \end{aligned} \tag{23}$$

Small values of c allow one to obtain the fourth- or fifth-order accurate boundary values of extrapolated grid functions practically independent of starting values at the corresponding opposite boundaries.

Compact quadrature formulas. Considering integrals over x between limits in the vicinity of x_j (for example, $\int_{x_j}^{x_{j+1}} u(\xi)d\xi$, $\int_{x_{j-1/2}}^{x_{j+1/2}} u(\xi)d\xi$, $\int_{x_{j-1}}^{x_j} u(\xi)d\xi$), the corresponding compact approximations can be constructed in the form of corrections to the standard two- or three-point operators. Though the resulting order can be higher than the initial one, there is no point to viewing them as alternatives to powerful high-order numerical integration tools like the Simpson and Gauss rules. However, they can serve as basis operators in the case of extremely high-order multioperators approximations.

Correcting, for example, the trapezoid rule for $[x_{j-1}, x_j]$, one can obtain the "left" operator

$$Q_l = \frac{1}{2}(I + T_{-1}) - \frac{1}{12}N_l\Delta_2$$

for which

$$Q_l(c)[u]_j = \frac{1}{h}\int_{x_{j-1}}^{x_j} u(\xi)d\xi - \frac{1}{24}(1+2c)h^3u_x^{(3)} + \frac{1}{360}(2 + 15c - 30c^2)h^4u_x^{(4)} + O(h^5).$$

Similarly, the expansion series for the "right" operator

$$Q_l = \frac{1}{2}(I + T_1) - \frac{1}{12}N_r\Delta_2$$

looks as

$$Q_r(c)[u]_j = \frac{1}{h}\int_{x_j}^{x_{j+1}} u(\xi)d\xi + \frac{1}{24}(1+2c)h^3u_x^{(3)} + \frac{1}{360}(2 + 15c - 30c^2)h^4u_x^{(4)} + O(h^5).$$

In the above example, the additive correction increases approximation order of the trapezoid rule. In the context of multioperators, it makes sense to consider also compact approximations which order is less than that of an initial formula. As an example, consider the following multiplicative correction to the Simpson rule.

$$Q_{1,l} = N_l(\frac{1}{3}(4I + T_1 + T_{-1}) - c(\Delta_0 - \Delta_2)),\ Q_{1,r} = N_r(\frac{1}{3}(4I + T_1 + T_{-1}) - c(\Delta_0 + \Delta_2).$$

It gives

$$Q_{1,l}(c)[u]_j = \frac{1}{h}\int_{x_{j-1}}^{x_{j+1}} u(\xi)d\xi - \frac{1}{3}ch^3u_x^{(3)} + \frac{1}{90}(1 + 15c + 30c^2)h^4u_x^{(4)} + O(h^5). \quad (24)$$

Thus the approximation order is not increased but one has now one-parameter family allowing to construct formally arbitrary-order multiopertors.

B. *Multioperators*

General forms.

We return to the one-parameter operators families defined by Eqs.(19) or (20) in the previous Section. Their general properties discussed therein show that the coefficients in the Taylor expansion series for the operators actions are polynomial of successively increased degrees in the parameter c. In particular, the above particular examples present the first two coefficients as the first and second degree polynomials in c. It suggests using operators from the families as basis operators admitting uniquely defined multioperators.

Denoting by L_l either of the two mth-order left operators, we fix M different values $c_1, c_2, \dots, c_M$ and construct the "left" linear combination

$$L_{M,l} = \sum_{i=1}^{M} \gamma_i^{(l)} L_l(c_i), \quad (25)$$

where coefficients $\gamma_i^{(l)},\ i = 1, 1, \dots, M$ is a partition of unity. To annihilate $M - 1$ expansion terms, the coefficients are required to satisfy the linear system

$$\sum_{i=1}^{M} \gamma_i^{(l)} = 1,\ \sum_{i=1}^{M} p_1(c_i)\gamma_i^{(l)} = 0,\ \sum_{i=1}^{M} p_2(c_i)\gamma_i^{(l)} = 0,\ \dots, \sum_{i=1}^{M} p_{M-1}(c_i)\gamma_i^{(l)} = 0, \quad (26)$$

where $p_k,\ k = 1, 2, \dots, M$ are kth-degree polynomials served as coefficients in the Taylor expansion series for $L_l(c_i)[u]_j$. Equalities (26) are the conditions of cancelling the terms with $h^m, h^{m+1}, \dots, h^{m+M-2}$ in the series. Successively extracting from Eqs. (26) equalities for the sums $\sum_{i=1}^{M} c_i\gamma_i^{(l)}, \sum_{i=1}^{M} c_i^2\gamma_i^{(l)}, \dots, \sum_{i=1}^{M} c_i^{M-1}\gamma_i^{(l)}$, one arrives at a system with the Vandermonde matrix

$$\sum_{i=1}^{M} \gamma_i^{(l)} = 1,\ \sum_{i=1}^{M} c_i\gamma_i^{(l)} = r_1,\ \sum_{i=1}^{M} c_i^2\gamma_i^{(l)} = r_2,\ \dots, \sum_{i=1}^{M} c_i^{M-1}\gamma_i^{(l)} = r_m, \quad (27)$$

where $r_1, r_2, \dots, r_{M-1}$ are numerical constants defined by the coefficients of $p_k(c)$. The similar system arises when constructing the "right" operator $L_{M,r}$ based on linear combinations $L_{M,r} = \sum_{i=1}^{M} \gamma_i^{(r)} L_r(c_i)$ where L_r is either of the two right operators. Based on the invertibility of the Vandermonde matrices, the following statement is true.

Theorem *Let $c_1, c_2, \dots, c_M$ be distinct real numbers while $\gamma_i^{(l)},\ \gamma_i^{(r)},\ i = 1, 2, \dots, M$ satisfy (27) with the right hand sides $r_i = r_i^{(l)}$ and $r_i = r_i^{(r)}$ respectively. Then there exist uniquely defined linear combinations $L_{M,l} = \sum_{i=1}^{M} \gamma_i^{(l)} L_l(c_i),\ L_{M,r} = \sum_{i=1}^{M} \gamma_i^{(r)} L_r(c_i)$ satisfying*

$$L_{M,l}[u]_j = [Lu]_j + O(h^{m+M-1}), \quad L_{M,r}[u]_j = [Lu]_j + O(h^{m+M-1})$$

for sufficiently smooth functions $u(x)$.

As evident from the expressions for the basis operators considered in the previous subsection, the "left" and the "right" ones differ only in signs of their either skew-symmetric or self-adjoint components. It means that the polynomial $p_k(c)$ in the corresponding Taylor expansion series are the same while the

TABLE III

ACCURACY OF MULTIOPERATORS QUADRATURE FORMULAS BASED ON THE MODIFICATION OF THE SIMPSON RULE

N	8	16	32	64
I_{simp}	5.81D-6	3.28D-7	2.05D-8	1.28D-9
I_5	2.38D-7	1.35D-8	4.77D-10	1.57D-11
k		4.13	4.82	4.92
I_7	2.18D-4	1.83D-8	7.99D-13	7.19D-15
k		13.5	14.5	6.79

series differ only in signs of their terms. Hence it follows that $r_i^{(l)} = r_i^{(r)},\ i = 1, 2, \ldots, M-1$. In turn, it leads to the equalities $\gamma_i^{(l)} = \gamma_i^{(r)}$. It can be easily proved that the left and right multioperators satisfy

$$L_{M,l} = -L^*_{M,r} \quad \text{if} \quad L_l = -L^*_r, \quad L_{M,l} = L^*_{M,r} \quad \text{if} \quad L_l = L^*_r$$

The operators $L_{M,l}$ $L_{M,r}$ will be referred to as left and right multioperator respectively. It should be emphasized that their actions on known grid functions can be calculated simultaneously and independently when using M parallel processors. As in the cases considered on Section 3, the coefficients $\gamma_i,\ i = 1, \pm 1, \pm 2, \ldots, M$ can be obtained in an analytical form. To reduce their absolute values increasing with M, the zeroes of the Mth-order Chebyshev polynomials for chosen intervals $(c_{min}, c_{max}),\ c_{min} > 0$ are supposed to be the c_i parameters. The lesser are values of $|c_i|$, the greater is the effect of decaying influence of boundary conditions in (18). However, the excessively small values may considerably increase condition numbers of (27).

Considering various grid functionals and starting from their two- or three-point conventional approximations, one can create formally arbitrary-order multioperators. In particular , the above compact approximations to integrals, interpolation and extrapolation operators can be used providing different r4ight hand sides of (27). As an illustration, we consider below multioperators approximations to integrals which take advantage of the decaying influence of boundary conditions.

Multioperators quadratures. We choose the corrections to the Simpson rule, that is $(Q_{1,l}, Q_{1,r})$ as the basis operators. From the Taylor expansion series (24) one can easily obtain the right hand sides of (27) in the case $M = 3$. They are $(1, 0, -1/30)$. Retaining more terms in (24), one can construct desired order approximations to $\int_{x_{j-1}}^{x_{j+1}} u(\xi)d\xi$ which upon summing give high-accuracy approximations to integrals. Consider an integration interval $[a, b]$. To take advantage of the decaying influence of boundary conditions, one can use left multioperators for the limits $[(b-a)/2, b]$ and right multioperators for the limits $[a, (b-a)/2]$ with the basis operators formulas as starting values in the procedures (18). For small c_i, the left multioperators do not practically "feel" the starting boundary values at $x = a$ while the right ones do not practically "feel" the starting boundary values at $x = b$

The calculations were carried out for the fifth- and seventh-order multioperators ($M = 3$ and $M = 5$) approximating $\int_0^1 \sin(\xi)d\xi$. They are denoted by I_5 and I_7 respectively. More or less arbitrary chosen parameters $(c_1 = .08, c_2 = .1, c_3 = .12)$ and $(c_{min} = .01, c_{max} = .1)$ were used for the fifth-order and the seventh-orders multioperators respectively. As boundary conditions, the Simpson formula applied to the intervals $[x_0, x_2]$ and $[x_{N-2}, x_N]$ was used.

In Table 3, the l_1 norms of the integration errors for the Simpson rule I_{simp} and the multioperators are presented for several numbers of nodes N. The corresponding mesh convergence orders estimates k are also included in the Table.

As seen, the fifth-order multioperator outperforms the sevenths-order one if $N = 8$ but the latter shows striking increase of accuracy when N increases.

V. CONCLUSIONS

The presented multioperators principle is a general way to construct desired-order approximations to various grid functionals. It can be applied to PDE's or any numerical analysis formula whose order is required to be increased. In the case of parallel machines, the increase of accuracy can be achieved simply by increasing numbers of processors without changing tasks for each processor.

In the case of fluid dynamics, it is shown that remarkably high accuracy and high resolution multioperators schemes can be constructed. Other presented numerical examples also show high efficiency of the approach thus illustrating its multipurpose nature.

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