When is it optimal to stop an advertisment campaign?

C. V. Nikolopoulos [*] A. Yannacopoulos [†]

Abstract

In this work we derive a model for product diffusion based on the Bass model and assuming that the effect of the advertisement campaign in the diffusion of the product is subject to a stochastic dependence. The resulting free boundary problem is solved numerically in order to find the policy that allows us to determine the optimal stopping time of the advertisement campaign of a product.

Keywords. Stochastic ODE, Free Boundary Problem.

1 The model

A very popular and succesful model for technology diffusion is the Bass model. According to the standard Bass model, the adoption of a new product is associated with two different procedures. The first is the adoption of the product because of the effect of advertisement. This is assumed to be a processs which may be modelled by a linear term, i.e. if $n(t)$ is the number of customers that have adopted the product up to time t, and N is the total market potential, then the rate of change of $n(t)$ with respect to time is likely to be proportional to the number of potential customers that have not yet adopted the product, $N - n(t)$. The proportionality factor a is a measure of the effectiveness of the advertisement campaign for the product. The second process that plays an important effect is the so called word of mouth process which assumes that the various potential adopters are affected by the peer pressure of those that have already adopted the product. This is a second order effect and the rate of change of adopters due to this mechanism is likely to be proportional to the number of adopters $n(t)$ multiplied by the number of potential adopters $N - n(t)$. Thus the rate of change of adopters due to this mechanism will be of the form $bn(t)(N - n(t))$. The proportionality factor B is a measure of the effectiveness of the word of mouth mechanism.

Therefore, the evolution of the number of customers that have adopted the product will follow the differential equation

$$\frac{dn}{dt} = A\,(N - n(t)) + B\,n(t)(N - n(t))$$

Working with the scaled variable $x(t) = n(t)/N$ the equation becomes

$$\frac{dx}{dt} = a\,(1 - x(t)) + b\,x(t)(1 - x(t))$$

The Bass model has been used widely to the modelling of the diffusion of a wide range of products, with very good fit to real data, and has been proved very successful as a model. However, it is a fairly simple model which has to be further enhanced with realistic effects so as to provide a better understanding and aproximation of the real world. One important step towards this direction is to introduce some uncertainty in the various parameters of the model. This uncertainty for the purpose of the present paper will be modeled with the use of white noise, however, there will be comments concerning the generalization to more realistic types of noise, e.g. noise terms including jumps etc. The noise will model uncertainty concerning the various model parameters, such as the effectiveness of the advertisement campaign or the effectiveness of the word of mouth mechanism. We will then assume that $a(t) = a_d(t) + a_s(t)$ where $a_d(t)$ is the deterministic (average) part of the parameter $a(t)$ and $a_s(t)$ is the stochastic part which models the fluctuations around the deterministic part. A reasonable first model for the fluctuating term would be to model them as a Gaussian process (on account of laws of the large numbers considerations) and a suitable model for that would be to take $a_s(t)$ to be given by an Itô integral.

Under these modelling assumptions, the Bass model will be formulated as a stochastic differential equation of the Itô type, and of the form

$$dx(t) = \{a\,(1 - x(t)) + b\,x(t)(1 - x(t))\}dt + \sigma_0 a(1 - x(t))dW(t)$$

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[*] Department of Mathematics, University of the Aegean, Karlovasi, Samos, Greece (cnikolo@aegean.gr).

[†] Department of Statistics, Athens University of Economics and Business, Athens, Greece (ayannaco@aegean.gr)

where by $W(t)$ we denote the Wiener process.

We now assume that the effectiveness of advertisment a can be controlled through an advertisement campaign for the product. Of course this is done at some cost. Let $c(a,t)$ be the cost of advertising the product for time t at intensity a. We shall consider the problem of determination of an optimal advertisement campaign for a product, such that the firm receives the maximum expected profit from the sales of the product but at the minimum possible cost. Therefore, this problem can be redressed as an optimal stopping problem, the solution of which will provide us with the optimal stopping time of the advertisement campaign.

Assume that the product starts at time s with sales $x(s) = x$. The cost functional of the problem, will be of the form

$$\Theta(s,x;t) = J(a) = E\left[\left(e^{-r(t-s)} f_0(x(t)) + \int_s^t e^{-r(t-s)} f(x(s^{'}))ds^{'}\right) - e^{-r(t-s)} c(a,t-s) \mid \mathcal{F}_s\right], \tag{1}$$

where by $\mathcal{F}_s = \sigma(W(u), u \leq s)$ we denote the $\sigma-$ algebra which is generated by the Wiener process. We may consider this $\sigma-$ algebra as containing the information about the history of the product up to time s. The term $e^{-r(t-s)}$ is a discount factor that allows us to compare values obtained at different times. The function f_0 is a profit function that gives us the profit obtained by sales $x(t)$ at time t. The second term, which is expressed as an integral gives us the intertemporal profit from all the sales of the product within the interval $[s,t]$. For the sake of simplicity in this paper we consider that $f_1(x) = 0$, i.e. the firm receives no intertemporal profit but is only interested in the final condition of the sales at time t. However, our results are easily generalized to the case where $f_1(x) \neq 0$. The functions $f_0(x)$ and $f_1(x)$ must be increasing functions of their argument.

From the form of the previous cost functional we see that as time proceeds and the advertising campaign is still active, the firm derives increasing profit from the increasing sales of the product. However, at all times she must take into account the increasing cost of the advertisement campaign. At some point, the cost of the campaign becomes so large that it is no longer worth pursuing it any more. This is the optimal stopping time for the campaign. The purpose of this paper is to derive the optimal time when the firm should stop advertising the product and leave the market.

2 The optimal stopping problem

We now formulate the optimal stopping problem

> Find a stopping time τ^* such that
>
> $$\phi(s,x) = \Theta(s,x;\tau^*) = sup_{\tau}\Theta(s,x;\tau)$$
>
> where the supremum is taken over all stopping times τ. Recall that a stopping time is a random time τ such that $\{\omega, \tau \leq t\} \in \mathcal{F}_t$.
> The stopping time τ^* will be the optimal stopping time for the advertisement campaign.

This problem will be solved through the formulation of a free boundary value problem.

Proposition 1. The optimal value of the cost functional $\phi(s,x)$ satisfies the differential inequality

$$\begin{aligned} \mathcal{A}\phi - r\phi = 0 \quad \text{in} \quad & D = \{(s,x) \in V, : \phi >= g\} \\ \text{with} \quad & \phi = g \quad \text{in}\, \partial D. \end{aligned} \tag{2}$$

where $\mathcal{A} = \frac{\partial}{\partial s} + \{a(1-x) + bx(1-x)\}\frac{\partial}{\partial x} + \sigma_0^2 a^2 (1-x)^2 \frac{\partial^2}{\partial x^2}$ and $g(s,x) = f_0(x) - c(a,s)$.
The optimal stopping time is the first time when the sales process $x(t)$ hits the boundary D.

This problem will have to be completed with a final (initial) condition and a boundary condition. The final condition will be of the from $\phi(T,x) = f_0(x)$ (we omit the advertisement cost term, since the advertisement campaign has had no time to run so it does not cost anything) whereas a reasonable choice for the boundary condition may be Neumann type boundary conditions at $x = 0$ and $x = 1$. Such types of conditions model the fact that for these limiting situations we will not have significant change in the profit by changing our initial position in the market x by a small amount.

3 Formulation of the numerical method

In order to treat this problem numerically we will present an iteration method, of a finite difference scheme, motivated by [3].

The equations to be solved, in their complete form, can be written in the following way

$$H(s,x,\phi,D_s\phi,D_x\phi,D_x^2\phi) = \min\left[r\phi - \tfrac{\partial\phi}{\partial s} - \alpha(x)\tfrac{\partial\phi}{\partial x} - \tfrac{\sigma_0^2}{2}\beta^2(x)\tfrac{\partial^2\phi}{\partial x^2},\ \phi(s,x) - g(s,x)\right] = 0, \tag{3}$$

$$\tfrac{\partial\phi}{\partial x} = 0, \quad \text{at } x = 0 \quad \text{and } x = 1, \quad s \in [0,T], \tag{4}$$

$$\phi(x,T) = f_0(x), \quad x \in [0,1], \tag{5}$$

where $\alpha(x) = a(1-x) + bx(1-x)$, $\beta(x) = a(1-x)$. In addition have that $g(x,s;a) = (f_0(x) - c(a,s))$.

Let $B([0,T]\times[0,1])$ denote the space of continuous bounded functions $\phi(s,x)$ on $[0,T]\times[0,1]$,(i.e. $\|\phi\|_\infty := \|\phi\| \leq M$ for some $M > 0$) and let $h > 0$ denote the spatial step and $k > 0$ the time step. We consider the following discrete approximations:

$$\frac{\partial\phi}{\partial s} \simeq \delta_s\phi(s,x), \quad \frac{\partial\phi}{\partial x} \simeq \delta_x\phi(s,x), \quad \frac{\partial^2\phi}{\partial x^2} \simeq \delta_x^2\phi(s,x)$$

Therefore the corresponding discrete version of equation (3), following a similar approach as in the problem studied in [3], is

$$\begin{aligned} \min\Bigg[&\phi(s,x)\left(r + \frac{1}{k} + \frac{\sigma_0^2\beta^2(x)}{h^2} + \frac{\alpha(x)}{h}\right) - \phi(s,x+h)\left(\frac{\sigma_0^2\beta^2(x)}{2h^2} + \frac{\alpha(x)}{h}\right) \\ &-\frac{1}{k}\phi(s+k,x) - \phi(s,x-h)\frac{\sigma_0^2\beta^2(x)}{2h^2},\ \phi(s,x) - g(s,x)\Bigg] = 0. \end{aligned} \tag{6}$$

Then we can define the mapping $S : \mathbb{R}^+ \times \mathbb{R}^+ \times [0,1] \times \mathbb{R} \times B([0,T]\times[0,1]) \to \mathbb{R}$

$$\begin{aligned} S(k,h,x,y,\phi) = \min\Big[&yh\left(r + \tfrac{1}{k} + \tfrac{\sigma_0^2\beta^2(x)}{h^2} + \tfrac{\alpha(x)}{h}\right) - \tfrac{h}{k}\phi(s+k,x) \\ &-\phi(s,x+h)\left(\tfrac{\sigma_0^2\beta^2(x)}{2h} + \alpha(x)\right) - \phi(s,x-h)\tfrac{\sigma_0^2\beta^2(x)}{2h},\ \phi(s,x) - g(s,x)\Big] = 0, \end{aligned}$$

for $x \in [h, 1-h]$, while for $x = 0, 1$, the mapping S, has a similar form, accounting for the boundary conditions.

Therefore equation (6) is equivalent to $S = 0$. In addition the coefficients of ϕ are all negative and hence S is monotone, i.e. for all $\phi, \psi \in B([0,T]\times[0,1])$, k, $h \in \mathbb{R}^*$, $x \in [0,1]$, $y \in \mathbb{R}$ we have for $\phi \geq \psi$,

$$S(k,h,x,y,\phi) \leq S(k,h,x,y,\psi).$$

The scheme S is said to be consistent if for every $x \in [0,1]$, $s \in [0,T]$ and for every $w(\cdot,\cdot) \in C^{1,2}([0,T]\times[0,1])$ we have

$$\lim_{z\to x,\, k\to 0,\, h\to 0,\, \epsilon\to 0} \frac{S(k,h,z,w(s,z)+\epsilon, w+\epsilon)}{h} = H(t,x,w,D_s w, D_x w, D_x^2 w).$$

We also need to show that the scheme is stable i.e. that there exist a bounded solution of the scheme $\phi_h \in B_M([0,T],[0,1])$ ($\|\phi_h\| \leq M$) to the equation $S(k,h,x,\phi(s,x),\phi) = 0$.

Lemma 1. *(i) The scheme S is consistent for $w(\cdot,\cdot) \in C^{1,2}([0,T]\times[0,1])$.*
(ii) The scheme $S$is stable.

Finally we may show that the scheme converges.

Proposition 2. *The numerical solution ϕ_h converges to the unique viscosity solution of equations (3-5) as $k \to 0$ and $h \to 0$.*

Note that the accuracy of this iterative scheme is of $O(k+h)$. More accurate schemes can be constructed by using explicit or implicit finite difference schemes.

4 Numerical Simulations

We will demonstrate the numerical solution of the problem by choosing an arbitrary example. The numerical solution is obtained by applying the iteration scheme (7). We assumed in the simulations that $c(a,s) = c_0(T-s)$

where the dependence of c_0 on a is assumed to be absorbed in the constant c_0 and that f_0 has a simple linear form $f_0 = \gamma x$, for γ a constant. Note that different forms of the functions $c(a,s)$ and $f_0(x)$ can also be considered.

We take $\sigma_0 = .5$, $r = .1$, $a = 1$, $b = 1$ and $T = 2$. We choose f_0, $f_0 = \gamma x$, for $\gamma = 0.9$, and for the cost function c, $c = c_0(T - s)$ for $c_0 = .4$. In figure (1) we plot the free boundary $q(s)$ by solving the problem using the iteration method (equation (7)). The iteration starts with initial value of $\phi_h^1 = 1$ and stopping the run when $\|\phi_h^{n-1} - \phi_h^n\| \leq \epsilon$ for ϵ being the accuracy we pose. Assuming that we have a random graph of the amount of sales (crooked line) the point (q^*, τ^*), $q^* = q(\tau^*)$, will indicate the optimal stopping time where the advertisement campaign should stop.

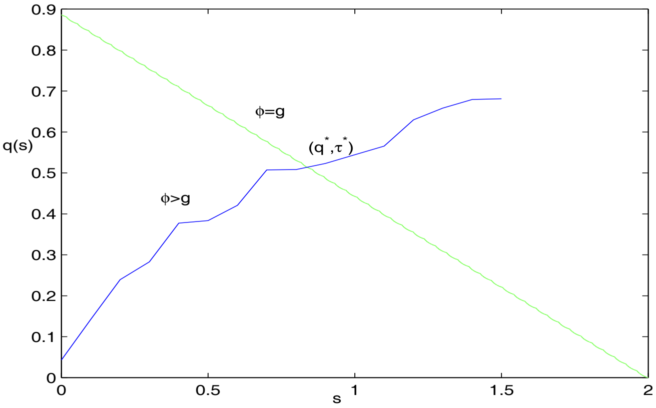

Figure 1: *Form of the free boundary, q(s), plotted against time, arising by the numerical solution of problem (3)- (5) for* $M_h = 81$, $h = .0125$, $k = 7.8 \cdot 10^{-5}$, *and accuracy* $\epsilon = 10^{-6}$.

We can notice that due to the linear form of the functions f_0 and $c(s)$ we have also a linear form in the graph of the free boundary $q(s)$.

We can have also a different choice of the cost function, e.g $c(s) = c_0(T - s)^p$, $0 < p < 1$ expressing the fact that allowing to advertise a product for longer times one can be able to achieve better prices and thus lower cost. Note also that an alternative choice would be $c(s) = c_0 \ln(T - s)$, etc. and that similar forms of dependence can be taken also for the function f_0. The choise of the cost function can be shown to have an effect on the free boundary.

References

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