Delay Analysis of Continuous Sampling Plans

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Abstract

In this paper we consider quality control inspection schemes for continuous production processes. In particular we analyze the delay behavior of an inspection station that operates according to Dodge's Continuous Sampling Plan. The analysis is carried out under the assumption that parts are produced according to a Poisson process, inspection times are exponential, and parts are independently defective with probability q. The model is analyzed using matrix geometric techniques. We also analyze a class of queuing models obtained from the examination of the post-inspection phase of the process. These are queueing models with arrival streams modulated by the inspection station and are also treated via matrix geometric methods.

1 Introduction

Continuous Sampling Plans (CSP) were proposed by Dodge [3] as means for on–line quality monitoring of continuous production processes where no natural production lots exist. The simplest such scheme (CSP-1) comprises two modes of operation. Mode I involves sampling at random the output process with selection probability $1/r$ (where $r > 1$) for each part. This sampling process continues until a defective part is found at which point the inspector switches to mode II which involves 100% inspection of the output process. The inspector remains in mode II until k consecutive good parts are found at which point he reverts to mode I (sampling with selection probability $1/r$). A markovian analysis of this sampling plan is discussed in section 2.

Since the contribution of the inspection time and its associated queueing delays, if any, to the total cycle time of the parts may in some cases not be negligible we examine explicitly the queueing aspects of this inspection process. This is carried out in section 4 where two related models are analyzed. In both models it is assumed that parts arrive according to a Poisson process with rate λ and that they are good or defective independently with probability p. They join an infinite capacity queue in front of the inspector. Inspection times are assumed independent, exponential random variables with rate μ. In the first model we will consider, which we term the *diverted stream model*, when the inspector switches to its sampling mode, each incoming part joins the queue with probability $1/r$ or is diverted downstream (not inspected) with probability $1 - 1/r$. In the second inspector model, which we call the *accelerated inspection model,* every part joins the inspector queue, even when the inspector is in sampling mode. However, when in sampling mode, the inspector works with rate $r\mu$ and detects defective parts with probability $1/r$. The performance of these models is again analyzed using matrix geometric techniques.

Finally, in section 5, we examine the effect of this inspection scheme on the variability of the downstream arrival process and in particular we analyze both a queueing system with input a Markov Arrival Process (MAP) formed by the accepted parts stream, and a system with a MAP input derived from the rejected parts. This second system models a repair station for the defective parts that are detected.

2 Markovian Description of the Inspection Policy

Dodge [3] proposed a number of Continuous Sampling Plans (CSP) for quality control of continuous production. The simplest of them, CSP-1 allows the inspector to operate in two modes which we will call the *exhaustive* and the *sampling* mode. When the inspector is in exhaustive mode, every single part produced is inspected and defective parts are rejected. When k consecutive good parts have been found the inspector switches to the sampling mode. In this mode each part is inspected with probability $1/r$ independently and parts that are not inspected are accepted. This continues until a defective part is found which causes the inspector to switch back to the exhaustive mode. We refer the reader to [11] and to [5], [6], and [10] for further details on CSP and its modifications such as multilevel sampling plans. In what follows we give a brief account of the markovian analysis for CSP under the assumption that produced parts are independently good with probability p or defective with probability $q = 1 - p$.

Consider a discrete time Markov chain with state space $S := \{0, 1, 2, \ldots, k\}$ and transition probability matrix

$$P = \begin{bmatrix} q & p & 0 & 0 & \cdots & 0 & 0 \\ q & 0 & p & 0 & \cdots & 0 & 0 \\ q & 0 & 0 & p & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ q & 0 & 0 & 0 & \cdots & 0 & p \\ q/r & 0 & 0 & 0 & \cdots & 0 & 1-q/r \end{bmatrix}. \tag{1}$$

This irreducible, aperiodic chain describes the operation of the inspection policy. Each arrival to the inspection station triggers a state transition. Defective parts drive the state to zero when the process is in states $\{0, 1, 2, \ldots, k-1\}$ while in state k this happens only with probability $1/r$. The stationary distribution is obtained by the solution of the system

$$\begin{aligned} \pi_i &= p\pi_{i-1}, \qquad\qquad i = 1, 2, \ldots, k-1, \\ \pi_k &= p\pi_{k-1} + (1 - q/r)\pi_k, \end{aligned}$$

together with the normalization condition, which gives

$$\begin{aligned} \pi_i &= \frac{qp^i}{1 + p^k(r-1)}, \qquad i = 0, 1, \ldots, k-1, \\ \pi_k &= \frac{rp^k}{1 + p^k(r-1)}. \end{aligned}$$

The fraction of parts that are inspected is then given by $f = 1 - \pi_k + \pi_k r^{-1}$, or

$$f = \frac{1}{1 + p^k(r-1)}. \tag{2}$$

The proportion of defective parts that go undetected as a fraction of all parts that are produced is given by $q(1-f)$. Finally, if defective parts that are detected are destroyed, the Average Outgoing Quality (AOQ) which is defined as the percentage of good parts after the inspection is given by

$$AOQ = \frac{p}{1 - f + fp}.$$

The above analysis is based on the assumption that defective parts constitute a Bernoulli process. Under such an assumption of course the particular sampling procedure has no advantage over ordinary Bernoulli sampling which inspects every part independently with probability of inspection f. However, if the overall defective probability q results from alternating periods between higher and lower rates of defectives CSP-1 would be more effective than simple Bernoulli sampling. Other sampling plans including CSP-2 and multilevel schemes (see [11], [5], and [6] i.a.) safeguard against "stray defectives" by not reverting immediately to 100% as soon as a defective is detected.

3 Results of matrix geometric analysis

In this section we mention briefly some standard results of matrix geometric analysis that we will need in the sequel. We refer the reader to the standard works of Neuts [7] and Latouche and Ramaswami [4] for proofs and further information. A continuous time Markov process with state space $S := \{(i,j) : i = 0,1,2,\ldots; j = 0,1,\ldots,k\}$ and generator

$$Q = \begin{bmatrix} B_0 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & A_2 & A_1 & A_0 \\ & & & & \end{bmatrix}. \tag{3}$$

where A_0, A_1, A_2, B_0, are $(k+1)\times(k+1)$ matrices is called Quasi-Birth-Death process (QBD). The square matrices A_m, $m = 0,1,2$, and B_0 must of course satisfy the obvious conditions in order for Q to be an actual generator, namely all the entries of A_0 and A_2 and all non–diagonal entries of A_1 and B_0 must be non–negative. In addition, if e is a column vector in $\mathbb{R}^{k+1}$ with all entries equal to 1 we should have $(A_0 + B_0)e = 0$ and $(A_0 + A_1 + A_2)e = 0$. We will further assume that the matrices A_m, $m = 0,1,2$, and B_0 are such that the generator Q is irreducible.

Theorem 1 *The process with infinitesimal generator Q is positive recurrent if and only if the minimal nonnegative solution R to the matrix equation*

$$A_0 + RA_1 + R^2 A_2 = 0, \tag{4}$$

has all its eigenvalues inside the unit disc and the finite system of equations

$$\begin{aligned} x_0(B_0 + RA_2) &= 0 \\ x_0(I - R)^{-1}e &= 1 \end{aligned} \tag{5}$$

has a unique positive solution x_0. Then the stationary probability corresponding to Q is given by

$$p_i = x_0 R^i. \tag{6}$$

In partitioned form we have $p = [p_0, p_1, p_2, \ldots]$ where p_i, $i = 1,2,\ldots,$ are row vectors with $k+1$ components, $p_i = (p_{i,0}, p_{i,1}, \ldots, p_{i,k})$, corresponding to the $k+1$ different states for the arrival process.

The next theorem gives a simple criterion that allows one to determine whether Q is positive recurrent. $\mathrm{sp}(R)$ denotes the spectral radius of the matrix R.

Theorem 2 *If the matrix $A : = A_0 + A_1 + A_2$ is irreducible, then $\mathrm{sp}(R) < 1$ if and only if*

$$\pi A_0 e < \pi A_2 e \tag{7}$$

where π is the stationary probability vector of A, i.e. the probability vector that satisfies $\pi A = 0$.

4 Queueing aspects of the inspector

In this section we examine the delay aspects of the inspection problem. We assume that parts arrive at the inspector according to a Poisson process with rate λ and that the are good with probability p or defective with probability $q = 1 - p$ independently of each other and of the arrival process. We will examine two models for the inspector. In the first, which we will term the *diverted stream model,* we assume that when the inspector is in the 100% inspection mode all parts join the inspector queue. However, when the inspector switches to the intermittent inspection mode then each arriving part independently joins the queue with probability $1/r$ or moves downstream with probability $1 - 1/r$. Once a part has joined the inspector queue it is inspected in a FIFO fashion by the inspector. Inspection times are assumed independent exponential random variables with rate μ. In the second model, which we will term the *accelerated inspection model,* all parts join the inspector queue, even when the inspector operates in its sampling mode. When the inspector operates in the 100% inspection mode inspection times are again exponential random variables with rate μ. However, when the inspector switches to the sampling mode, after k consecutive good parts have been detected, the inspection rate becomes $r\mu$ while at the same time a defective part which is inspected is detected as such with probability $1/r$. Both models are analyzed by means of matrix geometric techniques.

4.1 Diverted stream model

This model can be described by a continuous time Markov chain with state space $S := \{(i,j) : i = 0,1,2,\ldots, j = 0,1,\ldots,k\}$ where the first component, i, designates the number of customers in the system while the second, j, the phase in the markovian description of the inspector. Each time the inspector finds a defective part he immediately goes to phase 0. Each consecutive good part inspected causes the inspector phase to increase by one until he reaches phase k. There he remains until he inspects a defective part, at which point he switches to phase 0. When the inspector is in phase k the station switches to a regime of intermittent inspections. Thus it does not inspect every single part but instead every rth part. In fact, for modeling convenience, we will consider a modified version of this inspection policy whereby when the inspector is in the intermittent phase arriving parts join the inspection queue with probability r^{-1} independently. The generator of this Markov process, Q, is given by (3) where A_i, $i - 0,1,2$, and B_0 are $(k+1)\times(k+1)$ matrices given by

$$B_0 = \begin{bmatrix} -\lambda & & & \\ & -\lambda & & \\ & & \ddots & \\ & & & -\lambda/r \end{bmatrix}, \quad A_0 = \begin{bmatrix} \lambda & & & & \\ & \lambda & & & \\ & & \ddots & & \\ & & & \lambda & \\ & & & & \lambda/r \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -(\lambda+\mu) & & & \\ & -(\lambda+\mu) & & \\ & & \ddots & \\ & & & -(\lambda/r+\mu) \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mu q & \mu p & & & \\ \mu q & & \mu p & & \\ & & & \ddots & \\ \mu q & & & & \mu p \\ \mu q & & & & \mu p \end{bmatrix}$$

The stability condition for this model is $\pi A_0 e < \pi A_2 e$ (see equation 7) where π is the vector of stationary probability of the matrix $A = A_0 + A_1 + A_2$ which is given by

$$A = \mu \begin{bmatrix} -p & p & & & \\ q & -1 & p & & \\ & & \ddots & \ddots & \\ q & & & -1 & p \\ q & & & & -q \end{bmatrix}. \tag{8}$$

The solution of the equations $\pi A = 0$ together with the normalization condition $\sum_{i=0}^{k} \pi_i = 1$ gives

$$\begin{aligned} \pi_i &= p^i q, \qquad i = 0, 1, ..., k-1, \\ \pi_k &= p^k. \end{aligned} \tag{9}$$

Condition (7) becomes then

$$[\pi_0, \pi_1, ..., \pi_k] \begin{bmatrix} \lambda & & \\ & \lambda & \\ & & \lambda r^{-1} \end{bmatrix} \begin{bmatrix} 1 \\ \\ 1 \end{bmatrix} < [\pi_0, \pi_1, ..., \pi_k] \begin{bmatrix} \mu q & \mu p & \\ & & \mu p \\ \mu q & & \mu p \end{bmatrix} \begin{bmatrix} 1 \\ \\ 1 \end{bmatrix}$$

or

$$\lambda/\mu < \frac{1}{1 - p^k(1 - r^{-1})}. \tag{10}$$

4.2 Accelerated inspection model

Here we examine the second model. The generator is again given by (3) with

$$B_0 = \begin{bmatrix} -\lambda & & & \\ & -\lambda & & \\ & & \ddots & \\ & & & -\lambda \end{bmatrix}, \quad A_0 = \begin{bmatrix} \lambda & & & \\ & \lambda & & \\ & & \ddots & \\ & & & \lambda \end{bmatrix}$$

$$A_1 = \begin{bmatrix} -(\lambda+\mu) & & & \\ & -(\lambda+\mu) & & \\ & & \ddots & \\ & & & -(\lambda+r\mu) \end{bmatrix}, \quad A_2 = \begin{bmatrix} \mu q & \mu p & & & \\ \mu q & & \mu p & & \\ & & & \ddots & \\ \mu q & & & & \mu p \\ \mu q & & & & \mu(r-q) \end{bmatrix}$$

It is easy to check that the matrix $A = A_0 + A_1 + A_2$ is the same as (8) and hence that the corresponding equilibrium distribution π is given by (9). The stability condition $\pi A_0 e < \pi A_2 e$ gives then $\lambda < \mu(1 - \pi_k + \pi_k r)$ or

$$\lambda/\mu < 1 + p^k(r-1). \tag{11}$$

One can easily see that, for $0 < p < 1$, $\frac{1}{1-p^k(1-r^{-1})} < 1 + p^k(r-1)$ for all k and $r > 0$. Hence the stability region for the diverted steam model is smaller thatn that for the accelerated isnspection model. In this respect the latter appears to be a preferable inspection system.

For both models, once the matrix R defined in theorem 1 has been determined, once can compute various performance criteria of interest. For example, the expected number of parts in the inspection station are given by $L = \sum_{i=1}^{\infty} i p_i e = x \left(\sum_{i=1}^{\infty} i R^i\right) e$ where the infinite sum of matrices converges by virtue of the fact that $sp(R) < 1$. In view of the identity $\left(\sum_{i=1}^{\infty} i R^{i-1}\right)(I - R) = \sum_{i=0}^{\infty} R^i = I - R$ we have

$$L = x(I-R)^{-2} R\, e.$$

Corresponding expected delays can be obtained from Little's law.

5 The downstream queue

In this section we examine the modulating effect of the CSP inspector under the assumption that the inspection occurs at the final stage of a production process. The post–inspection stream replenishes a make–to–stock system which satisfies a Poisson demand with rate λ. The an intermediate stage of the production process. In such a situation one of the effects of the inspection process is that it increases the variability of the arrival process. The main point is on the parts as they move downstream stream of parts. Roughly speaking the main point is In this model we consider as

negligibly small the time of parts in the inspector. After inspection defective parts leaving the system while good parts join a queue. The discipline in the queue is FIFO and time is exponential with rate μ. The state space of the queueing process is $S := \{(i,l) : i = 0, 1, 2, ...; j = 0, 1, ..., k\}$ where the first component, i, designates the number of customers in the queue while the second, j, the phase of the inspector.

$$B_0 = \begin{bmatrix} -\lambda p & & & \\ & -\lambda p & & \\ & & \ddots & \\ & & & -\lambda(1-q/r) \end{bmatrix} \qquad A_0 = \begin{bmatrix} 0 & \lambda p & & & \\ & 0 & \lambda p & & \\ & & \ddots & \ddots & \\ & & & 0 & \lambda p \\ & & & & \lambda(1-q/r) \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \mu & & & \\ & \mu & & \\ & & \ddots & \\ & & & \mu \end{bmatrix} \quad A_1 = \begin{bmatrix} -(\lambda p+\mu) & & & & \\ \lambda q & -(\lambda+\mu) & & & \\ \vdots & & \ddots & & \\ \lambda q & & & -(\lambda+\mu) & \\ \lambda q/r & & & & -(\lambda+\mu) \end{bmatrix}$$

The stability condition can once again be obtained from (7). In this model

$$A = \begin{bmatrix} -\lambda p & \lambda p & & \\ \lambda q & -\lambda & & \\ & & & \\ \lambda q r^{-1} & & & -\lambda q r^{-1} \end{bmatrix}$$

and the solution of the equations $\pi A = 0$ is

$$\begin{aligned} \pi_i &= \frac{p^i q}{p^k(r-1)+1}, \qquad i = 0, 1, \ldots, k-1, \\ \pi_k &= \frac{p^k r}{p^k(r-1)+1}. \end{aligned}$$

Thus (7) in this case gives $\lambda p + \lambda q(1 - r^{-1})\pi_k < \mu$ or

$$\lambda p < \mu \frac{1+(r-1)p^k}{1+(r-1)p^{k-1}}.$$

5.1 The repair station for defective parts

In this section we consider the defective parts that are detected by the inspection policy and assume that they are transferred to a repair station. Repair times are assumed to be exponential with rate μ. The arrival stream to the repair station consists of the defective parts that are detected by the inspector operating under the CSP-1 policy. The state space of the queueing process is $S := \{(i,j) : i = 0, 1, 2, ...; j = 0, 1, ..., k\}$ where the first component, i, designates the number of customers in the queue while the second, j, the phase of the inspector. The generator, Q, has again the QBD form given by (3) with

$$B_0 = \begin{bmatrix} -\lambda q & & & \\ & -\lambda q & & \\ & & \ddots & \\ & & & -\lambda q r^{-1} \end{bmatrix}, \quad A_0 = \begin{bmatrix} \lambda q & & 0 \\ \lambda q & & \\ \vdots & & \\ \lambda q r^{-1} & & 0 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \mu & & & \\ & \mu & & \\ & & \ddots & \\ & & & \mu \end{bmatrix}, \quad A_1 = \begin{bmatrix} -(\lambda+\mu) & \lambda p & & & \\ & -(\lambda+\mu) & \lambda p & & \\ & & \ddots & \ddots & \\ & & & -(\lambda+\mu) & \lambda p \\ & & & & -(\lambda q r^{-1}+\mu) \end{bmatrix}$$

The stability condition is obtained by the same methodology with

$$A = \begin{bmatrix} -\lambda p & \lambda p & & \\ \lambda q & -\lambda & & \\ & & \ddots & \\ \lambda q r^{-1} & & & -\lambda q r^{-1} \end{bmatrix}$$

which is the same as the matrix for model 2. So, the probability vector is the same as before. The condition is $\pi A_0 e < \pi A_2 e$ gives $\lambda[q(1-\pi_k) + \pi_k q r^{-1}] < \mu$ or

$$\lambda q < \mu \left(1 + p^k(r-1)\right).$$

The delay analysis for these systems can again be carried out using the proceedure outlined in section 4.

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