

A NEW START-UP DEMONSTRATION TEST

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Abstract In this paper, we introduce and study a new acceptance / rejection rule suitable for use in start-up demonstration tests. According to this rule, the equipment is accepted if k consecutive successes occur, while the equipment is rejected if two failures separated by at most $r-2$ successes are observed. General formulae and analytic expressions are given for the probability mass function, the mean and the variance of the number of start-ups until the termination of the test. The distribution of the number of start-ups until the termination of the test with acceptance or rejection is also deduced. Finally, three inferential procedures for the estimation of the unknown probability of successful start-up are discussed.

Index Terms-- Start-up demonstration test, Scan Statistics, Markov chain embedding technique, EM algorithm

I. INTRODUCTION

A start-up demonstration test is a procedure which elucidates the reliability of an equipment with regard to its ability in performing successful start-ups. In the early research on start-up demonstration testing, as discussed by Hahn and Gage [7], the equipment is accepted as soon as a prespecified number of consecutive (k) successful start-ups occurs in a series of attempted start-ups of the equipment (consecutive successes scheme, or briefly CS scheme). They considered the case where the successive start-ups constitute a sequence of iid Bernoulli trials with success probability p (probability of a successful start-up). Viveros and Balakrishnan [13] derived further results for the CS scheme and discussed some inferential aspects for the unknown probability p of a successful start-up.

Balakrishnan and Chan [3] introduced and studied a modification of the CS scheme which is as follows: If k consecutive successful start-ups are achieved before the appearance of f in total unsuccessful start-ups, then the equipment under test is accepted; if f in total unsuccessful start-ups are achieved before the appearance of k consecutive successful start-ups, then the equipment under test is rejected (consecutive successes total failures scheme, or briefly CSTF scheme). Additional results and for the CSTF have been established by Smith and Griffith [12] and Martin [10].

In this paper we introduce and study the consecutive successes distance failures start-up demonstration test (or briefly CSDF scheme) which is as follows: if k consecutive successful start-ups achieved before the appearance of two unsuccessful start-ups separated by at most $r-2$ successful ones the equipment is accepted; if two unsuccessful start-ups separated by at most $r-2$ successful start-ups are achieved before the appearance of k consecutive successful start-ups then the equipment is rejected ($k \geq r \geq 2$). The main difference between the CSTF and the CSDF scheme is that in the latter we deem it very important for the acceptance of the equipment that unsuccessful start-ups take place in a long distance controlled by the parameter r .

In Section 2 we give all the necessary notations related to the following three waiting time random variables: Total number of trials till the termination of the test, number of trials till the termination of the test because k consecutive successful start-ups are observed and number of trials till the termination of the test due to the occurrence of two unsuccessful start-ups separated by at most $r-2$ successful ones. All three variables are studied in Section 3 and the main quantities of interest for assessing the performance of a CSDF start-up demonstration test are derived. In Section 4 three inferential procedures based on maximum likelihood estimation methods for the estimation of p are discussed.

II. DEFINITIONS AND NOTATIONS

Let $\{X_n, n \geq 1\}$ be a sequence of iid Bernoulli random variables taking on the values 1 (success) and 0 (failure) with probabilities $p = \Pr(X_i = 1)$ and $q = \Pr(X_i = 0) = 1 - p$ respectively. Denote by E_1 the event corresponding to the occurrence of a success run of length k , and by E_0 the event corresponding to the occurrence of two failures which are separated by at most $r-2$ successes ($r \geq 2, k \geq r$). Let T be the waiting time until E_1 or E_0 occurs, whichever comes sooner, i.e. T is the waiting time for the occurrence of the compound pattern

$$E = E_1 \cup E_0 = \underbrace{\{11\dots 1\}}_k \cup \{00, 010, 0110, \dots, 0\underbrace{11\dots 10}_{r-2}\}.$$

The probability mass function (pmf), the cumulative distribution function (cdf) and the probability generating function (pgf) of T will be respectively denoted by

$$f_T(n) = \Pr(T = n), F_T(n) = \Pr(T \leq n), G_T(s) = \sum_{n=0}^{\infty} f_T(n)s^n.$$

Next, we denote by T_1 (resp. T_0) the waiting time for E which is due to the occurrence of event E_1 (resp. E_0). For the waiting time T_i , $i = 0, 1$, we will use analogous notations to those of T , that is

$$f_{T_i}(n) = \Pr(T_i = n), F_{T_i}(n) = \Pr(T_i \leq n), G_{T_i}(s) = \sum_{n=0}^{\infty} f_{T_i}(n)s^n.$$

Let us now consider an equipment which is subject to a CSDF start-up demonstration test and denote by p the probability of a successful start-up in an attempted trial. Then the probability of arriving at a decision (acceptance or rejection of the equipment) exactly at the n -th trial, equals $f_T(n)$. The first term of the RHS of the expression $f_T(n) = f_{T_1}(n) + f_{T_0}(n)$ accounts for the probability of acceptance of the equipment at the n -th trial, while the second one, to the probability of rejection of it.

III. MAIN RESULTS

In the present section, the basic quantities for the waiting time random variable T introduced in Section 2 are derived using a finite Markov chain embedding technique similar to the one employed by Koutras and Alexandrou [9] (see, also Fu [5], Antzoulakos [1], Koutras [8] and Fu and Lou [6]).

To start with, we define a Markov chain $\{Y_t, t \geq 1\}$ with finite state space $\Omega = \{1, 2, \dots, k + r - 1\}$ operating on the block of iid trials X_1, X_2, \dots, X_t as follows:

- (i) If $X_1 = X_2 = \dots = X_t = 1$ for $1 \leq t \leq k - 1$ we assign to Y_t the value t ;
- (ii) If $X_t = 0$ and $X_{t-1} = X_{t-2} = \dots = X_1 = 1$ for $1 \leq t \leq k$ we assign to Y_t the value k ;
- (iii) If $X_t = 0$ and $X_{t-1} = X_{t-2} = \dots = X_{t-i} = 1$ for $r - 1 \leq i \leq k - 1$ we assign to Y_t the value k ;
- (iv) If $X_t = X_{t-1} = \dots = X_{t-i+1} = 1$ and $X_{t-i} = 0$ for $1 \leq i \leq r - 2$ we assign to Y_t the value $k + i$;
- (v) If $X_t = X_{t-1} = \dots = X_{t-i+1} = 1$ and $X_{t-i} = 0$ for $r - 1 \leq i \leq k - 1$ we assign to Y_t the value i .

In the above definitions we assume that, before the end of the block of trials we are working with, neither the event E_0 nor the event E_1 has occurred. Finally, we complete our state space

by appending the state $k+r-1$, which serves as an absorbing state and accounts for all the configurations where at least one of the events E_0, E_1 has been observed.

It is easy to check that the above definitions establish a homogeneous Markov chain $\{Y_t, t \geq 1\}$ on Ω , with initial probability vector

$$\boldsymbol{\pi} = [\Pr(Y_1 = 1), \Pr(Y_1 = 2), \dots, \Pr(Y_1 = k+r-1)] = p\mathbf{e}_1 + q\mathbf{e}_k$$

($\mathbf{e}_i, i = 1, 2, \dots, k+r-1$, denotes the unit (row) vector of \mathbb{O}^{k+r-1} having 1 at the i -th coordinate and 0 elsewhere) and transition probability matrix \mathbf{P} given by

$$\mathbf{P} = \left[\begin{array}{c|c} \mathbf{Q} & (\mathbf{I}-\mathbf{Q})\mathbf{1}' \\ \hline \mathbf{0} & 1 \end{array} \right]_{(k+r-1) \times (k+r-1)}$$

where

$$\mathbf{Q} = \left[\begin{array}{c|c} \mathbf{Q}_{11} & \mathbf{Q}_{12} \\ \hline \mathbf{Q}_{21} & \mathbf{Q}_{22} \end{array} \right]_{(k+r-2) \times (k+r-2)}, \quad \mathbf{Q}_{11} = \begin{bmatrix} 0 & p & 0 & \cdots & 0 \\ 0 & 0 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(k-1) \times (k-1)},$$

$$\mathbf{Q}_{12} = \begin{bmatrix} q & 0 & \cdots & 0 \\ q & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ q & 0 & \cdots & 0 \end{bmatrix}_{(k-1) \times (r-1)}, \quad \mathbf{Q}_{22} = \begin{bmatrix} 0 & p & 0 & \cdots & 0 \\ 0 & 0 & p & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & p \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(r-1) \times (r-1)},$$

matrix \mathbf{Q}_{21} is an $(r-1) \times (k-1)$ matrix with all its entries vanishing except for the entry $(r-1, r-1)$ which equals p , $\mathbf{0}$ and $\mathbf{1}$ are $1 \times (k+r-2)$ (row) vectors with all their entries 0 and 1, respectively, and \mathbf{I} is the identity $(k+r-2) \times (k+r-2)$ matrix.

Starting with the identity $\Pr(T \leq n) = \Pr(Y_n = k+r-1)$ and making use of

$$\mathbf{P}^n = \left[\begin{array}{c|c} \mathbf{Q}^n & (\mathbf{I}-\mathbf{Q}^n)\mathbf{1}' \\ \hline \mathbf{0} & 1 \end{array} \right], \quad n \geq 0$$

it may be checked that

$$F_T(n) = \Pr(T \leq n) = \boldsymbol{\alpha}(\mathbf{I}-\mathbf{Q}^{n-1})\mathbf{1}', \quad n \geq 1,$$

$$f_T(n) = P(T = n) = \begin{cases} 0, & n = 1 \\ \boldsymbol{\alpha} \mathbf{Q}^{n-2} (\mathbf{I} - \mathbf{Q}) \mathbf{1}', & n \geq 2, \end{cases}$$

$$G_T(s) = \sum_{n=0}^{\infty} f_T(n) s^n = s(1 - (1-s)\boldsymbol{\alpha}(\mathbf{I} - s\mathbf{Q})^{-1}\mathbf{1}') = s^2 \boldsymbol{\alpha} (\mathbf{I} - s\mathbf{Q})^{-1} (\mathbf{I} - \mathbf{Q}) \mathbf{1}'$$

$$\mu'_{[m]} = E[T(T-1)\cdots(T-m+1)] = \begin{cases} 1 + \boldsymbol{\alpha} (\mathbf{I} - \mathbf{Q})^{-1} \mathbf{1}', & m = 1 \\ m! \boldsymbol{\alpha} (\mathbf{I} - \mathbf{Q})^{-m} \mathbf{Q}^{m-2} \mathbf{1}', & m \geq 2, \end{cases}$$

where $\boldsymbol{\alpha} = [\Pr(Y_1 = 1), \Pr(Y_1 = 2), \dots, \Pr(Y_1 = k + r - 2)] = p\mathbf{e}_1 + q\mathbf{e}_k$.

Carrying out some algebra we may express the expected number $E(T) = \mu'_{[1]}$ of start-ups until the termination of the test, as

$$E(T) = \frac{(1-p^k)(2-p^{r-1})}{q(1+p^k-p^{r-1})}.$$

It can be easily verified that for fixed values of p and r (k) the mean of T is an increasing (decreasing) function of k (r).

The investigation of the distribution of the waiting time random variable T_1 may be performed via a proper modification of the transition probability matrix \mathbf{P} employed in the foregone analysis. To achieve that, we modify the Markov chain used for the study of T , by canceling the transitions to the absorbing state $k+r-1$ for all the configurations leading to the occurrence of the event E_0 (and modifying appropriately the respective entries of the transition probability matrix). The updated transition probability matrix now reads

$$\mathbf{P}_1 = \left[\begin{array}{c|c} \mathbf{Q} & (\mathbf{I} - \mathbf{Q} - \mathbf{M}_1) \mathbf{1}' \\ \hline \mathbf{0} & 1 \end{array} \right]_{(k+r-1) \times (k+r-1)}$$

where \mathbf{M}_1 is a $(k+r-2) \times (k+r-2)$ matrix with all its entries being zero except for the last $r-1$ entries of its diagonal which equal q . On a similar manner, the cdf, the pmf and the pgf of T_1 are respectively given by

$$F_{T_1}(n) = \boldsymbol{\alpha} (\mathbf{I} - \mathbf{Q}^{n-1}) (\mathbf{I} - \mathbf{Q})^{-1} (\mathbf{I} - \mathbf{Q} - \mathbf{M}_1) \mathbf{1}', \quad n \geq 1,$$

$$f_{T_1}(n) = \begin{cases} 0, & n = 1 \\ \boldsymbol{\alpha} \mathbf{Q}^{n-2} (\mathbf{I} - \mathbf{Q} - \mathbf{M}_1) \mathbf{1}', & n \geq 2, \end{cases}$$

$$G_{T_1}(s) = s^2 \mathbf{a}(\mathbf{I} - s\mathbf{Q})^{-1}(\mathbf{I} - \mathbf{Q} - \mathbf{M}_1)\mathbf{1}'.$$

Likewise, we study the waiting time T_0 by simply substituting \mathbf{M}_1 with \mathbf{M}_0 , a $(k+r-2) \times (k+r-2)$ matrix which entries are all 0 except for the entry $(k-1, k-1)$ which equals p .

IV. STATISTICAL INFERENCE FOR THE PROBABILITY OF SUCCESSFUL START-UP

Suppose that n start-up demonstration tests are performed, denote by N_i , S_i and F_i the number of attempts, successes and failures respectively, until the termination of the i -th test ($1 \leq i \leq n$) and set

$$S = \sum_{i=1}^n S_i, \quad F = \sum_{i=1}^n F_i, \quad N = \sum_{i=1}^n N_i = S + F.$$

Then, (see Viveros and Balakrishnan [13]), the maximum likelihood estimator (MLE) of p is given by $\hat{p} = S/N$ while the observed Fisher information takes on the form

$$I(\hat{p}) = \left. \frac{\partial^2 l(p)}{\partial p^2} \right|_{p=\hat{p}} = \frac{N}{\hat{p}(1-\hat{p})}.$$

Thus, an approximate (asymptotic) $(1-a)\%$ confidence interval for p will be given by $\hat{p} \pm z_{a/2} / \sqrt{I(\hat{p})}$. We briefly refer to this method as the VB method.

Smith and Griffith [12] proposed a different approach for estimating of p in cases where only the number of different configurations in which the n start-up demonstration tests have been terminated (We briefly refer to this method as the SG method). In our case we have r different configurations for the termination of the test, namely

$$C_1 = \{\underbrace{11\dots 1}_k\}, \quad C_{0,1} = \{00\}, \dots, \quad C_{0,r-1} = \{0\underbrace{11\dots 10}_{r-2}\}.$$

Applying a suitable Markov chain embedding technique we obtain that the probabilities p_{C_1} , $p_{C_{0,1}}$, \dots , $p_{C_{0,r-1}}$ that the test terminates on the occurrence of configuration C_1 , $C_{0,1}$, \dots , $C_{0,r-1}$, respectively, are given by

$$p_{C_1} = \frac{p^k(2-p^{r-1})}{1-p^{r-1}+p^k}, \quad p_{C_{0,i}} = \frac{p^{i-2}(1-p)(1-p^k)}{1-p^{r-1}+p^k}, \quad i=1,2,\dots,r-1.$$

Manifestly, if the configurations $C_1, C_{0,1}, \dots, C_{0,r-1}$ have been observed $n_1, n_{0,1}, \dots, n_{0,r-1}$ times, respectively, then the likelihood function will be given by

$$L(p) \propto p_{C_1}^{n_1} \prod_{i=1}^{r-1} p_{C_{0,i}}^{n_{0,i}}$$

and the MLE of p will follow from the solution of the equation $\partial \ln(L(p)) / \partial p = 0$ (via numerical methods). An approximate confidence intervals for p can also be established, by applying large sample maximum likelihood methods, as $\hat{p} \pm z_{\alpha/2} / \sqrt{I(\hat{p})}$.

A third scenario arises when only the number of trials until termination of each one of the n tests is available. Since the mean of the total number of attempts until termination of the test is not a monotone function of p the method of moments cannot be applied here. In such cases, the EM algorithm (see, e.g., Robert and Casella [11]) may offer an estimation procedure for the probability of a successful start-up (see Chan, Ng and Balakrishnan [4]).

The log likelihood function $l(p; F, N)$ is of the form

$$l(p; F, N) = N \log p + F \log((1-p)/p),$$

the expected complete-data log likelihood function will read

$$N \log p + \left(\sum_{i=1}^n E(F | N = N_i) \right) \log \left(\frac{1-p}{p} \right)$$

and (on differentiating it with respect to p and setting the outcome equal to zero) the following EM iterative scheme ensues

$$p^{(j+1)} = 1 - \frac{\sum_{i=1}^n E(F | N = N_i, p^{(j)})}{N}$$

($p^{(j)}$ denotes the current estimate of p). The determination of the conditional expected value $E(F | N = N_i, p^{(j)})$ may be performed through the Markov chain embedding methodology established in Chadjiconstantinidis, Antzoulakos and Koutras [2]. Furthermore, the observed Fisher information, which can be exploited for constructing approximate confidence intervals for p , takes on the form

$$I(\hat{p}) = \frac{\partial^2 l(p; F, N)}{\partial p^2} \Big|_{p=\hat{p}} = E \left(\frac{\partial^2}{\partial p^2} l(p; F, N) \Big| N \right)_{p=\hat{p}} + \text{Var} \left(\frac{\partial}{\partial p} l(p; F, N) \Big| N \right)_{p=\hat{p}}.$$

From our extensive numerical experimentation we conclude that best point and interval estimate is provided by the VB method, as it was expected, since this approach makes use (and requires) much more information (data) than the other two methods. Also, the standard deviation of the estimate deduced by the VB method is uniformly better than the one derived by the other two methods, despite the fact that the point estimate produced by it is not always the best one. This does not hold true for the other two methods comparing them to each other since for a range of p values the SG approach is proved superior while for the rest, the EM algorithm performs better.

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REFERENCES

- [1]. Antzoulakos, D.L., 2001. Waiting times for patterns in a sequence of multistate trials. *Journal of Applied Probability* 38, 508-518.
 - [2]. Chadjiconstantinidis, S., Antzoulakos, D.L., Koutras, M.V., 2000. Joint distributions of successes, failures and patterns in enumeration problems. *Advances in Applied Probability* 32, 866-884.
 - [3]. Balakrishnan, N., Chan, P.S., 2000. Start-up demonstration tests with rejection of units upon observing d failures. *Annals of the Institute of Statistical Mathematics* 52, 184-196.
 - [4]. Chan, P.S., Ng, T.H.K., Balakrishnan, N., 2006. Statistical inference for start-up demonstration tests with rejection of units upon observing d failures. Submitted for publication.
 - [5]. Fu, J.C., 1996. Distribution theory of runs and patterns associated with a sequence of multi-state trials. *Statistica Sinica* 6, 957-974.
 - [6]. Fu, J.C. Lou, W.Y.W., 2003. *Distribution Theory of Runs and Patterns and Its Applications*. World Scientific, New Jersey.
 - [7]. Hahn, G.J., Gage, J.B., 1983. Evaluation of start-up demonstration test. *Journal of Quality Technology* 15, 103-106.
 - [8]. Koutras, M.V. 2003. Application of Markov chains to the distribution theory of runs and patterns. In: Shanbhag, D.N., Rao, R.C. (Eds), *Handbook of Statistics*, Vol. 21, Elsevier Science B.V., pp. 431-472.
 - [9]. Koutras, M.V., Alexandrou, V.A. 1997. Sooner waiting time problems in a sequence of trinary trials. *Journal of Applied Probability* 34, 593-609.
 - [10]. Martin, D.E.K., 2004. Markovian start-up demonstration tests with rejection of units upon observing d failures. *European Journal of Operational Research* 155, 474-486.
 - [11]. Robert, P.C., Casella, G., 2005. *Monte Carlo Statistical Methods*, 2nd ed. Springer, New York.
 - [12]. Smith, M.L.P., Griffith S.W., 2005. Start-up demonstration tests based on consecutive successes and total failures. *Journal of Quality Technology* 37, 186-198.
 - [13]. Viveros, R., Balakrishnan, N., 1993. Statistical inference from start-up demonstration test data. *Journal of Quality Technology* 25, 119-130.
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