Tail Gini's Risk Measures and Related Linear Programming Models for Portfolio Optimization

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Abstract

Several polyhedral risk measures have been recently introduced leading to Linear Programming (LP) models for portfolio optimization. In this paper we study LP solvable portfolio optimization models based on the tail Gini's mean difference risk measurement. We use combinations of the Conditional Value at Risk (CVaR) measures to get some approximations to the tail Gini's mean difference with the advantage of being computationally much simpler than the Gini's measure itself. We introduce the weighted CVaR (WCVaR) measures defined as simple combinations of a very few CVaR measures with specific type of weights settings which relates the WCVaR measure to the tail Gini's mean difference. This allows us to use a few tolerance levels as only parameters specifying the entire WCVaR measures while the corresponding weights are automatically predefined by the requirements of the corresponding tail Gini's measure. All the studied models are SSD consistent and LP computable. We analyze both the theoretical properties of the models and their performances on the real-life data.

Index Terms

Portfolio optimization, linear programming, risk measures, stochastic dominance, Conditional Value at Risk, Gini's mean difference.

I. Introduction

FOLLOWING Markowitz [4], the portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem where the expected return is maximized and some (scalar) risk measure is minimized. In the original Markowitz model the risk is measured by the standard deviation or variance. Several other risk measures have been later considered thus creating the entire family of mean-risk (Markowitz-type) models. In particular, the polyhedral risk measures have been introduced which leads to Linear Programming (LP) solvable models in the case of discrete random variables, i.e., in the case of returns defined by their realizations under specified scenarios. The LP solvability is very important for applications to real-life financial decisions where the constructed portfolios have to meet numerous side constraints (including the minimum transaction lots, transaction costs and mutual funds characteristics). The introduction of these features leads to mixed integer LP problems.

The simplest polyhedral risk measures are dispersion measures similar to the variance. Yitzhaki [17] introduced the mean-risk model using Gini's mean (absolute) difference as the risk measure. The Gini's mean difference turn out to be special aggregation techniques of the multiple criteria LP model [6] based on the pointwise comparison of the absolute Lorenz curves. The latter leads the quantile shortfall risk measures which are more commonly used and accepted. Recently, the second order quantile risk measures have been introduced in different ways by many authors [5], [11]. The measure, usually called the Conditional Value at Risk (CVaR) or Tail VaR, represents the mean shortfall at a specified confidence level. It leads

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to LP solvable portfolio optimization models in the case of discrete random variables represented by their realizations under specified scenarios. The CVaR measures maximization is consistent with the second degree stochastic dominance [8]. Several empirical analyses confirm its applicability to various financial optimization problems. Thus, the CVaR models seem to overstep the measure of Value-at-Risk (VaR) defined as the maximum loss at a specified confidence level which is commonly used in banking.

Although any CVaR measure is risk relevant, it represents only the mean within a part (tail) of the distribution of returns. Therefore, such a single criterion is in some manner crude for modeling various risk aversion preferences. The Gini's mean difference model combines all CVaR measures averaging all shortfalls. In order to enrich the modeling capabilities, one needs to treat differently some more or less extreme events. This require some techniques to enhance the downside risk aversion [1]. Alternatively, the Tail GMD measures may be applied [9], which averages the shortfall within specified quantiles. The latter may be approximated with appropriate combinations of multiple CVaR measures. In this paper we study such LP solvable portfolio optimization models based on the use of multiple CVaR measures. Our analysis has been focused on the Weighted CVaR measures defined as simple combinations of a very few CVaR measures. This allows us to use a few tolerance levels as only parameters specifying the entire WCVaR measures while the corresponding weights are automatically predefined by the requirements of the Tail Gini's measures. Theoretical properties and computational efficiency of the models is studied as well as their practical achievements are tested. All the studied models are shown to be SSD consistent.

II. Portfolio optimization and CVaR measures

Our analysis is focused on the portfolio optimization problem following the original Markowitz' formulation and is based on a single period model of investment. At the beginning of a period, an investor allocates the capital among various securities, thus assigning a nonnegative weight (share of the capital) to each security. Let $J = \{1, 2, \dots, n\}$ denote a set of securities considered for an investment. For each security $j \in J$, its rate of return is represented by a random variable R_j with a given mean $\mu_j = \mathbb{E}\{R_j\}$. Further, let $\mathbf{x} = (x_j)_{j=1,2,\dots,n}$ denote a vector of decision variables x_j expressing the weights defining a portfolio. To represent a portfolio, the weights must satisfy a set of constraints that form a feasible set $\mathcal{P}$. The simplest way of defining a feasible set is by a requirement that the weights must sum to one and short sales are not allowed, i.e.

$$\mathcal{P} = \{\mathbf{x} \ : \ \sum_{j=1}^{n} x_j = 1, \quad x_j \geq 0 \quad \text{for } j = 1, \dots, n\} \tag{1}$$

Hereafter, it is assumed that $\mathcal{P}$ is a general LP feasible set given in a canonical form as a system of linear equations with nonnegative variables: This allows one to include upper bounds on single shares as well as several more complex portfolio structure restrictions which may be faced by a real-life investor.

Each portfolio $\mathbf{x}$ defines a corresponding random variable $R_{\mathbf{x}} = \sum_{j=1}^{n} R_j x_j$ that represents the portfolio rate of return. We consider T scenarios with probabilities p_t (where $t = 1, \dots, T$). We assume that for each random variable R_j its realization r_{jt} under the scenario t is known. Typically, the realizations are derived from historical data treating T historical periods as equally probable scenarios ($p_t = 1/T$). The realizations of the portfolio return $R_{\mathbf{x}}$ are given as $y_t = \sum_{j=1}^{n} r_{jt} x_j$ and the expected value can be computed as $\mu(\mathbf{x}) = \sum_{t=1}^{T} y_t p_t = \sum_{t=1}^{T} \left[\sum_{j=1}^{n} r_{jt} x_j\right] p_t$. Similarly, several risk measures can be LP computable with respect to the realizations y_t.

The portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem where the mean $\mu(\mathbf{x})$ is maximized and the risk measure $\varrho(\mathbf{x})$ is minimized. In the original Markowitz model, the standard deviation was used as the risk measure. Several other risk measures have been later considered thus creating the entire family of mean-risk models (see [2] and [3]). These risk measures, similar to the standard deviation, are not affected by any shift of the outcome scale and are equal to 0 in the case of a risk-free portfolio while taking positive values for any risky portfolio. Unfortunately, such risk measures are not consistent with the stochastic dominance order [16] or other axiomatic models of risk-averse preferences.

In stochastic dominance, uncertain returns (modeled as random variables) are compared by pointwise comparison of some performance functions constructed from their distribution functions. The first performance function $F^{(1)}_{\mathbf{x}}$ is defined as the right-continuous cumulative distribution function: $F^{(1)}_{\mathbf{x}}(\eta) = F_{\mathbf{x}}(\eta) = \mathbb{P}\{R_{\mathbf{x}} \leq \eta\}$ and it defines the first degree stochastic dominance (FSD). The second function is derived from the first as $F^{(2)}_{\mathbf{x}}(\eta) = \int_{-\infty}^{\eta} F_{\mathbf{x}}(\xi)\ d\xi$ and it defines the second degree stochastic dominance (SSD). We say that portfolio $\mathbf{x}'$ *dominates* $\mathbf{x}''$ *under the SSD* ($R_{\mathbf{x}'} \succ_{_{SSD}} R_{\mathbf{x}''}$), if $F^{(2)}_{\mathbf{x}'}(\eta) \leq F^{(2)}_{\mathbf{x}''}(\eta)$ for all η, with at least one strict inequality. A feasible portfolio $\mathbf{x}^0 \in \mathcal{P}$ is called *SSD efficient* if there is no $\mathbf{x} \in \mathcal{P}$ such that $R_{\mathbf{x}} \succ_{_{SSD}} R_{\mathbf{x}^0}$.

Several portfolio performance measures were introduced as safety measures to be maximized, like the worst realization, analyzed by Young [18], and the CVaR risk measures we consider further. Opposite to risk measures, the safety measures may be consistent with formal models of risk-averse preferences [12], [16]. Actually, for any risk measure $\varrho(\mathbf{x})$ a corresponding *safety* measure $\mu_\varrho(\mathbf{x}) = \mu(\mathbf{x}) - \varrho(\mathbf{x})$ can be defined and viceversa [2], [3]. Note that while risk measure $\varrho(\mathbf{x})$ is a convex function of $\mathbf{x}$, the corresponding safety measure $\mu_\varrho(\mathbf{x})$ is concave. We say that the safety measure $\mu_\varrho(\mathbf{x})$ is *SSD consistent* (or that the risk measure $\varrho(\mathbf{x})$ is *SSD safety consistent*) if $R_{\mathbf{x}'} \succeq_{_{SSD}} R_{\mathbf{x}''}$ implies $\mu_\varrho(\mathbf{x}') \geq \mu_\varrho(\mathbf{x}'')$. If the safety measure is SSD consistent, then except for portfolios with identical values of $\mu(\mathbf{x})$ and $\mu_\varrho(\mathbf{x})$ (and thereby $\varrho(\mathbf{x})$), every efficient solution of the bicriteria problem

$$\max\{[\mu(\mathbf{x}), \mu_\varrho(\mathbf{x})] : \quad \mathbf{x} \in \mathcal{P}\} \tag{2}$$

is an SSD efficient portfolio [7]. Therefore, we will focus on the mean-safety bicriteria optimization (2) rather than on the classical mean-risk model.

Stochastic dominance relates the notion of risk to a possible failure of achieving some targets. Note that function $F^{(2)}_{\mathbf{x}}$, used to define the SSD relation, can also be presented as follows [7]: $F^{(2)}_{\mathbf{x}}(\eta) = \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\}$ and thereby its values are LP computable for returns represented by their realizations y_t. In this paper we focus on quantile shortfall risk measures related to the so-called *Absolute Lorenz Curves* (ALC) [5], [13] which represent the second quantile functions defined as

$$F^{(-2)}_{\mathbf{x}}(p) = \int_0^p F^{(-1)}_{\mathbf{x}}(\alpha) d\alpha \quad \text{for } 0 < p \leq 1 \quad \text{and} \quad F^{(-2)}_{\mathbf{x}}(0) = 0, \tag{3}$$

where $F^{(-1)}_{\mathbf{x}}(p) = \inf\ \{\eta : F_{\mathbf{x}}(\eta) \geq p\}$ is the left-continuous inverse of the cumulative distribution function $F_{\mathbf{x}}$. Actually, the pointwise comparison of ALCs provides an alternative characterization of the SSD relation [8] and

$$F^{(-2)}_{\mathbf{x}}(\beta) = \max_{\eta \in R}\ \left[\beta\eta - F^{(2)}_{\mathbf{x}}(\eta)\right] = \max_{\eta \in R}\ [\beta\eta - \mathbb{E}\{\max\{\eta - R_{\mathbf{x}}, 0\}\}] \tag{4}$$

where η is a real variable taking the value of β-quantile $Q_\beta(\mathbf{x})$ at the optimum.

For any real tolerance level $0 < \beta \leq 1$, the normalized value of the ALC defined as $M_\beta(\mathbf{x}) = F^{(-2)}_{\mathbf{x}}(\beta)/\beta$ is called the *Conditional Value-at-Risk (CVaR)*. The corresponding risk measure $\Delta_\beta(\mathbf{x}) = \mu(\mathbf{x}) - M_\beta(\mathbf{x})$ is called hereafter the (worst) *conditional semideviation*. Note that, for any $0 < \beta < 1$, the CVaR measures defined by $F^{(-2)}(\beta)$, opposite to below-target mean deviations $F^{(2)}(\eta)$, are risk relevant. They are also SSD consistent [8]. For a discrete random variable represented by its realizations y_t problem (4) becomes an LP. Thus

$$M_\beta(\mathbf{x}) = \max\ \left[\eta - \frac{1}{\beta}\sum_{t=1}^{T} d_t^- p_t\right] \quad \text{s.t.} \quad d_t^- \geq \eta - y_t, \quad d_t^- \geq 0 \quad \text{for } t = 1, \ldots, T. \tag{5}$$

Yitzhaki [17] introduced the GMD model using Gini's mean (absolute) difference as risk measure. For a discrete random variable represented by its realizations y_t, the *Gini's mean difference* $\Gamma(\mathbf{x}) =$

$\frac{1}{2}\sum_{t'=1}^{T}\sum_{t''=1}^{T}|y_{t'}-y_{t''}|p_{t'}p_{t''}$ is LP computable (when minimized). Yitzhaki [17] suggested to use the corresponding safety measure

$$\mu_{\Gamma}(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma(\mathbf{x}) = \mathbb{E}\{R_{\mathbf{x}} \wedge R_{\mathbf{x}}\} \tag{6}$$

to take advantages of its SSD consistency and LP computability.

Both the Gini's mean difference and the CVaR measures are related to the ALC (3). One may notice that $\Delta_\beta(\mathbf{x}) = \frac{1}{\beta}(\mu(\mathbf{x})\beta - F^{(-2)}_{\mathbf{x}}(\beta))$ while the Gini's mean difference may be expressed as $\Gamma(\mathbf{x}) = 2\int_0^1(\mu(\mathbf{x})\alpha - F^{(-2)}_{\mathbf{x}}(\alpha))d\alpha = 2\int_0^1 \alpha\Delta_\alpha(\mathbf{x})d\alpha$. Hence, the GMD safety measure summarizes all the CVaR measures as $\mu_{\Gamma}(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma(\mathbf{x}) = 2\int_0^1 F^{(-2)}_{\mathbf{x}}(\alpha)d\alpha = 2\int_0^1 \alpha M_\alpha(\mathbf{x})d\alpha$. Therefore, the stronger SSD consistency results have been recently shown for the GMD model [8], i.e., $R_{\mathbf{x}'} \succ_{_{SSD}} R_{\mathbf{x}''}$ implies $\mu_\Gamma(\mathbf{x}') > \mu_\Gamma(\mathbf{x}'')$ which guarantees that every efficient solution of the bicriteria problem (2) is an SSD efficient portfolio. On the other hand, its computational LP model (even when simplified by taking its LP dual [1]) requires T^2 variables. which makes it much more complicated than the CVaR model (5) using only T variables. In the next sections we will demonstrate that models based on a few CVaR criteria offer a very good compromise between the computationally complex GMD model and simplified CVaR.

III. Tail Gini's and weighted CVaR measures

In order to model downside risk aversion, instead of the Gini's mean difference, the *tail Gini's* measure [8], [9] is used. It is defined for any $\beta \in (0,1]$ by averaging the vertical diameters $d_p(\mathbf{x})$ within the tail interval $p \le \beta$ as:

$$\Gamma_\beta(\mathbf{x}) = \frac{2}{\beta^2}\int_0^\beta (\mu(\mathbf{x})\alpha - F^{(-2)}_{\mathbf{x}}(\alpha))d\alpha = \frac{2}{\beta^2}\int_0^\beta \alpha\Delta_\alpha(\mathbf{x})d\alpha. \tag{7}$$

For any $0 < \beta \le 1$, the tail Gini's measure $\Gamma_\beta(\mathbf{x})$ is SSD safety consistent. One may notice that the corresponding safety measure $\mu_{\Gamma_\beta}(\mathbf{x}) = \mu(\mathbf{x}) - \Gamma_\beta(\mathbf{x})$ can be expressed as

$$\mu_{\Gamma_\beta}(\mathbf{x}) = \mu(\mathbf{x}) - \frac{2}{\beta^2}\int_0^\beta (\mu(\mathbf{x})\alpha - F^{(-2)}_{\mathbf{x}}(\alpha))d\alpha = \frac{2}{\beta^2}\int_0^\beta F^{(-2)}_{\mathbf{x}}(\alpha)d\alpha$$

which allows us to consider it as a second degree CVaR measure.

In the simplest case of equally probable T scenarios with $p_t = 1/T$ (historical data for T periods), the tail Gini's measure for $\beta = K/T$ may be expressed as the weighted conditional semideviation $\Delta^{(K)}_{\mathbf{w}}(\mathbf{x})$ with tolerance levels $\beta_k = k/T$ for $k = 1, 2, \ldots, K$ and properly defined weights [9]. In a general case, we may resort to an approximation based on some reasonably chosen grid β_k, $k = 1, \ldots, m$ and weights w_k expressing the corresponding trapezoidal approximation of the integral in the formula (7). This leads us to the Weighted CVaR (WCVaR) measure defined as

$$M^{(m)}_{\mathbf{w}}(\mathbf{x}) = \sum_{k=1}^{m} w_k M_{\beta_k}(\mathbf{x}), \quad \sum_{k=1}^{m} w_k = 1, \quad w_k > 0 \quad \text{for } k = 1, \ldots, m \tag{8}$$

The WCVaR measure is, obviously, a safety measure and it is risk relevant in the sense that for any risky portfolio its value is less than that for the risk-free portfolio with the same expected return. The corresponding risk measure turns out to be the weighted sum of the $\Delta_{\beta_k}(\mathbf{x})$ measures thus forming the weighted conditional semideviation:

$$\Delta^{(m)}_{\mathbf{w}}(\mathbf{x}) = \mu(\mathbf{x}) - M^{(m)}_{\mathbf{w}}(\mathbf{x}) = \sum_{k=1}^{m} w_k \Delta_{\beta_k}(\mathbf{x}), \qquad \sum_{k=1}^{m} w_k \le 1, \quad w_k > 0 \quad \text{for } k = 1, \ldots, m \tag{9}$$

The latter is not affected by any shift of the outcome scale and it is equal to 0 in the case of a risk-free portfolio while taking positive value for any risky portfolio, thus representing a translation invariant and risk relevant dispersion parameter. Therefore, we are eligible to consider the corresponding Markowitz-type model and its mean-safety formalization (2):

$$\max\{[\mu(\mathbf{x}), M^{(m)}_{\mathbf{w}}(\mathbf{x})] : \mathbf{x} \in \mathcal{P}\} = \max\{[\mu(\mathbf{x}), \mu(\mathbf{x}) - \Delta^{(m)}_{\mathbf{w}}(\mathbf{x})] : \mathbf{x} \in \mathcal{P}\} \tag{10}$$

Since the CVaR measures are SSD consistent [9], the same applies to the WCVaR measure. Actually, the following assertion is valid.

Theorem 1: For any set of levels $0 < \beta_1 < \beta_2 < \ldots < \beta_m \leq 1$, except for portfolios with identical values of $\mu(\mathbf{x})$ and all conditional semideviations $\Delta_{\beta_k}(\mathbf{x})$, every efficient solution of the bicriteria problem (10) is an SSD efficient portfolio.

Proof: Let $\mathbf{x}^0 \in \mathcal{P}$ be an efficient solution to the bicriteria mean-safety model (10). Suppose that there exists $\mathbf{x}' \in \mathcal{P}$ such that $R_{\mathbf{x}'} \succ_{_{SSD}} R_{\mathbf{x}^0}$. Then, due to SSD consistency, $\mu(\mathbf{x}') \geq \mu(\mathbf{x}^0)$ and $M_{\beta_k}(\mathbf{x}') \geq M_{\beta_k}(\mathbf{x}^0)$ for all $k = 1, \ldots, m$. The latter together with the fact that $\mathbf{x}^0$ is efficient, implies that $\mu(\mathbf{x}') = \mu(\mathbf{x}^0)$ and $\sum_{k=1}^m w_k M_{\beta_k}(\mathbf{x}') = \sum_{k=1}^m w_k M_{\beta_k}(\mathbf{x}^0)$. Hence, $M_{\beta_k}(\mathbf{x}') = M_{\beta_k}(\mathbf{x}^0)$ for $k = 1, \ldots, m$, and therefore, $\Delta_{\beta_k}(\mathbf{x}') = \Delta_{\beta_k}(\mathbf{x}^0)$ for all $k = 1, \ldots, m$, which completes the proof. ■

Exactly, for any $0 < \beta \leq 1$, while using the grid of m tolerance levels $0 < \beta_1 < \ldots < \beta_k < \ldots < \beta_m = \beta$ one may define weights:

$$w_k = \frac{(\beta_{k+1} - \beta_{k-1})\beta_k}{\beta^2}, \quad \text{for } k = 1, \ldots, m-1, \quad \text{and} \quad w_m = \frac{(\beta_m - \beta_{m-1})\beta_m}{\beta^2} \tag{11}$$

where $\beta_0 = 0$. This results in the weighted sum $\sum_{k=1}^m w_k \Delta_{\beta_k}(\mathbf{x})$ expressing the trapezoidal approximation to the tail Gini's measure (7). Note that $\sum_{k=1}^m w_k = \beta_m^2/\beta^2 = 1$ and thus we get a regular weighted conditional semideviation (9) $\Delta^{(m)}_{\mathbf{w}}(\mathbf{x}) \cong \Gamma_\beta(\mathbf{x})$. Further, weights (11) generate a WCVaR measure (8) such that $M^{(m)}_{\mathbf{w}}(\mathbf{x}) \cong \mu_{\Gamma_\beta}(\mathbf{x})$.

We emphasize that despite being only an approximation to (7), any WCVaR measure with weights defined by (11) itself is a well defined LP computable measure with guaranteed SSD consistency in the sense of Theorem 1. Hence, it needs not to be built on a very dense grid to provide proper modeling of risk averse preferences. While using the uniform grid of levels $\beta_k = (k\beta)/m$ for $k = 1, 2, \ldots, m$ and gets weights defined as $w_k = (2k)/m^2$ for $k = 1, 2, \ldots, m-1$ and $w_m = 1/m$.

The commonly accepted approach to implementation of the Markowitz-type mean-risk models is based on the use of a specified lower bound μ_0 on expected return while minimizing the risk criterion. In our analysis we use the bounding approach applied to the maximization of the safety measures, i.e.

$$\max\{\mu_\varrho(\mathbf{x}) : \quad \mathbf{x} \in \mathcal{P}, \quad \mu(\mathbf{x}) \geq \mu_0\}. \tag{12}$$

For small values of the bound μ_0, the constraint $\mu(\mathbf{x}) \geq \mu_0$ does not influence the optimization (12). In this case, the portfolio obtained is the so called Maximum Safety Portfolio (MSP). In our analysis we have used the bounding approach (12) applied to the maximization of the WCVaR measures. For returns represented by their realizations we get an LP optimization problem:

$$\begin{aligned}
\text{maximize} \quad & \sum_{k=1}^m w_k q_k - \sum_{k=1}^m \frac{w_k}{\beta_k} \sum_{t=1}^T p_t d_{tk} \\
\text{subject to} \quad & \mathbf{x} \in \mathcal{P} \text{ and } \sum_{j=1}^n \mu_j x_j \geq \mu_0 \\
& d_{tk} - q_k + \sum_{j=1}^n r_{jt} x_j \geq 0, \ d_{tk} \geq 0 \quad \text{for } t = 1, \ldots, T; \ k = 1, \ldots, m
\end{aligned} \tag{13}$$

where q_k (for $k = 1, \ldots, m$) are unbounded variables taking the values of the corresponding β_k-quantiles (in the optimal solution). Except from the core portfolio constraints (1), model (13) contains T nonnegative variables d_{tk} and T corresponding linear inequalities for each k. Hence, its dimensionality is proportional to the number of scenarios T and to the number of tolerance levels m. Exactly, the LP model contains $m \times T + n$ variables and $m \times T + 2$ constraints. It does not cause any computational difficulties for a few hundreds of scenarios and a few tolerance levels, as in our computational analysis based on historical data. However, in the case of more advanced simulation models employed for scenario generation one may get several thousands of scenarios. This may lead to the LP model (13) with huge number of variables and constraints thus decreasing the computational efficiency of the model. If the core portfolio constraints contain only linear relations, like (1), then the computational efficiency can easily be achieved by taking advantages of the LP dual to model (13). The LP dual model takes the following form:

$$\begin{array}{ll} \text{minimize} & \eta - \mu_0 \xi \\ \text{subject to} & \eta - \mu_j \xi - \sum_{t=1}^{T} r_{jt} \sum_{k=1}^{m} u_{tk} \geq 0 \quad \text{for } j = 1, \ldots, n \\ & \sum_{t=1}^{T} u_{tk} \geq w_k \quad \text{for } k = 1, \ldots, m \\ & \xi \geq 0, \quad 0 \leq u_{tk} \leq p_t w_k / \beta_k \quad \text{for } t = 1, \ldots, T;\ k = 1, \ldots, m \end{array} \tag{14}$$

The dual LP model contains $m \times T$ variables u_{tk}, but the $m \times T$ constraints corresponding to variables d_{tk} from (13) take the form of simple upper bounds (SUB) on u_{tk} thus not affecting the problem complexity. Actually, the number of constraints in (14) is proportional to the total of portfolio size n and the number of tolerance levels m, thus it is independent from the number of scenarios. Exactly, there are $m \times T + 2$ variables and $m + n$ constraints. This guarantees a high computational efficiency of the dual model even for vary large number of scenarios.

IV. Experimental analysis

In our computational analysis we examine the MSPs for the different tested models. The analysis is performed on historical data are represented by weekly rates of return obtained by using stock prices from Milan Stock Exchange. The rates are computed as relative price variations. The data set consists of 157 securities quoted with continuity. The historical period covers six years during which the Italian Stock Exchange has shown alternate short periods of up and down trends. A set of 7 instances has been created, each of which takes into account the complete set of securities over a different time period. For each instance the Maximum Safety Portfolio (MSP) has been obtained through the use of the various tested models. In this section we only summarize and comment the main figures out of the huge amount of computational results we obtained.

The model based on the safety measure corresponding to the Gini's mean difference, i.e. the mean worse return, is referred simply as GMD. The CVaR model associated to a given tolerance level β is identified as CVaR(β). We have tested the CVaR model for five different values of β, i.e. CVaR(0.05), CVaR(0.1), CVaR(0.25) and CVaR(0.5). All the CVaR and the weighted CVaR models have been formulated according to (13). We have also tested two Tail WCVaR models:

- Model WCVaR(TG2) with two tolerance levels $\beta_1 = 0.1$, $\beta_2 = 0.25$ and weights $w_1 = 0.4$ and $w_2 = 0.6$.
- Model WCVaR(TG3) with three tolerance levels $\beta_1 = 0.1$, $\beta_2 = 0.25$, $\beta_3 = 0.5$ and weights $w_1 = 0.1$, $w_2 = 0.4$ and $w_3 = 0.5$.

While analyzing for all the models over all the periods, the diversification of the optimal portfolios (MSPs), one may notice that, in general, all the models have resulted in diversified portfolios. The number of selected securities for the Minimax model ranges between 6 and 29 securities. The GMD model has generated portfolios of 12 to 26 securities. The basic CVaR models as well as the WCVaR models provide portfolios similarly well diversified (14-30 securities).

TABLE I

OUT-OF-SAMPLE RESULTS ON MSPS: SINGLE PERIOD RETURNS.

Max. safety models	Periods 1	2	3	4	5	6	7	r_{min}	r_{max}	r_{med}	r_{av}
Minimax	39.77	85.43	348.50	-24.86	-60.31	0.99	10.33	-60.31	348.50	10.33	57.12
CVaR(0.05)	39.77	85.43	348.50	-24.86	-49.44	-2.84	9.51	-49.44	348.50	9.51	58.01
CVaR(0.1)	39.23	78.16	352.58	-25.94	-48.17	-7.46	39.95	-48.17	352.58	39.23	61.19
CVaR(0.25)	20.16	71.22	392.11	10.28	-54.66	31.95	34.90	-54.66	392.11	31.95	72.28
CVaR(0.5)	10.14	58.21	434.77	7.51	-55.21	39.74	58.90	-55.21	434.77	39.74	79.15
GMD	9.75	36.80	431.47	10.76	-53.08	47.40	10.33	-53.08	431.47	10.76	70.49
WCVaR(TG2)	28.27	71.23	385.45	-17.94	-47.15	10.67	140.90	-47.15	385.45	28.27	81.63
WCVaR(TG3)	21.41	52.27	404.90	7.78	-53.40	29.80	66.31	-53.40	404.90	29.80	75.58

We have also analyzed the models performances with respect to a long-run portfolio management. Each of the portfolios selected by a specific model in the 7 instances has been evaluated ex-post in the three months period following the date of selection. Table I provides the single period returns for each model expressed on a yearly basis. It is worth noticing that single period ex-post returns quite perfectly represent the upward and downward movements of the market. For instance, the high returns of all the models in period 3 can be partially interpreted as a consequence of the positive trend of the market at the beginning of the 1998 with a high positive jump of MIB30 performances in March. Similarly, negative results showed by all the models in the periods 5 are mainly due to the negative trend of the market in August 1998. To describe better out-of-sample results we have included into Table I also the following ex-post parameters: the minimum, average and maximum portfolio return (r_{min}, r_{av} and r_{max}, respectively); the median (r_{med}) of the average returns; Such performance criteria have been computed for all the models over all the periods and can be used to compare the out-of-sample behavior of the maximum safety portfolios selected by the different models.

TABLE II

OUT-OF-SAMPLE RESULTS ON MSPS: CUMULATIVE RETURNS.

Max. safety models	Periods 1	1-2	1-3	1-4	1-5	1-6	1-7
Minimax	39.77	60.99	115.24	71.91	28.23	23.22	22.83
CVaR(0.05)	39.77	60.99	115.24	71.91	34.59	27.47	24.74
CVaR(0.1)	39.23	57.50	112.92	69.81	33.93	25.93	27.84
CVaR(0.25)	20.16	43.44	106.15	78.28	38.32	37.24	36.90
CVaR(0.5)	10.14	32.00	100.87	77.91	35.02	35.79	38.88
GMD	9.75	22.53	91.36	72.42	32.90	35.21	31.34
WCVaR(TG2)	28.27	48.21	109.51	71.99	35.84	31.28	43.17
WCVaR(TG3)	21.41	35.97	100.98	78.10	36.20	35.12	39.19

Further, we cumulated the returns over the horizon up to 7 periods (21 months) to better analyze each model achievements. The figures shown in Table II are the cumulative returns of the portfolios selected by each model. Each column of the table refers to a period and provides the cumulative return of the portfolios selected over the preceding periods. For better understanding of the figures let us consider the first line of Table II which refers to the model Minimax. Each of the 7 portfolios selected by the Minimax model in the 7 instances has been evaluated ex-post in the three months investment period following the date of its selection. We define as $r_1, r_2, ..., r_7,$ the ex-post returns of these 7 portfolios. Then, the first column of Table II gives the ex-post return (after 3 months) of the first portfolio selected, i.e. r_1 (notice that such value is identical in Tables II and I). The second column of Table II gives the cumulative return of the portfolio selected in the first period and then modified after three months with the portfolio selected in the second period: the value is computed as $(1 + r_1)(1 + r_2) - 1$ where r_1 is the ex-post quarterly return for the first portfolio and r_2 is the ex-post quarterly return for the second one. Similarly, for all the other columns of the table; in particular, the last one provides the cumulated return over all the 7 periods.

These results have been computed to simulate a multi-period setting where, at no transaction cost, the portfolio changes over time. Rates are expressed on a yearly basis.

In Table II, it can be noticed that except for the Minimax and the extremal CVaR models ($\beta = 0.05$ or $\beta = 0.1$) all the other models resulted in similar cumulative return over the entire horizon of 21 months with (annual) rate of return exceeding 30%. The Minimax and the extremal CVaR models ($\beta = 0.05$ or $\beta = 0.1$) perform much worse than all the other models. Note that both the Tail WCVaR models and the CVaR(0.5) here have the best cumulative performances. It is interesting to notice that Also the GMD model is outperformed by simple Tail WCVaR models and the CVaR models for larger tolerance levels.

V. Concluding remarks

In this paper we have studied LP solvable portfolio optimization models based on extensions of the Conditional Value at Risk (CVaR) measure. The models use multiple CVaR measures thus allowing for more detailed risk aversion modeling. All the studied models are SSD consistent and may be considered some approximations to the Gini's mean difference with the advantage of being computationally much simpler than the GMD model itself. Our analysis has been focused on the weighted CVaR measures defined as simple combinations of a very few CVaR measures. We have introduced the specific type of weights settings which relate the WCVaR measure to the tail Gini's mean difference. This allows us to use a few tolerance levels as only parameters specifying the entire WCVaR measures while the corresponding weights are automatically predefined by the requirements of the corresponding tail Gini's measure.

Our experimental analysis of the models performance on the real-life data from the Milan Stock Exchange has confirmed their attractiveness. The weighted CVaR models have usually performed better than the GMD itself, the Minimax or the extremal CVaR models. These promising results show a need for further comprehensive experimental studies analyzing practical performances of the weighted CVaR models within specific areas of financial applications. It is important to notice that although the quantile risk measures (VaR and CVaR) were introduced in banking as extreme risk measures for very small tolerance levels (like $\beta = 0.05$), for portfolio optimization good results have been provided by rather larger tolerance levels.

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