NEWTON'S METHOD WITHOUT DIRECT FUNCTION EVALUATIONS

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Abstract

In this paper, we deal with the problem of solving systems of nonlinear equations. Most of the existed methods facing this problem require precise function and derivative values. So, it is useful to develop methods to work well when this information is not available or it is computationally expensive. To contribute in this area, we introduce a new strategy based on the replacement of function values in Newton's method by approximated quantities making Newton's method directly free of function evaluations and thus, ideal for imprecise function problems. From another point of view, it may be faced as an application of Newton's method on a new approximated system, equivalent to the original one. This approach is based on proper selected points, named pivot points, which are low-cost as they are extracted via a componentwise sign-function-based technique. The quadratic convergence of new algorithm is proven and the results are promising.

Index Terms

Newton's method, nonlinear systems, imprecise function values, pivot points, quadratic convergence, approximated systems.

I. INTRODUCTION

Consider the system of nonlinear equations

$$F(x) = 0 \tag{1}$$

where $F = (f_1, \ldots, f_n) : \mathcal{D} \subset \mathcal{R}^n \to \mathcal{R}^n$ is continuously differentiable on an open neighborhood $\mathcal{D}^* \subset \mathcal{D}$ of a solution $x^* = (x_1^*, \ldots, x_n^*) \in \mathcal{D}$ of the system.

There are several methods for solving such nonlinear systems [3], [8], [10]. Among them Newton's method is the most famous, because of its quadratic convergence when a good initial guess exists and the Jacobian matrix is nonsingular. Given an approximation x^p to the solution x^*, Newton's method computes the next approximation x^{p+1}, by

$$x^{p+1} = x^p - F'(x^p)^{-1}F(x^p), \quad p = 0, 1, \ldots$$

where $F'(x^p)$ is the Jacobian matrix of F, at the point x^p.

Most of the existed methods require precise function and derivative values. It is known that Newton's method is problematic when the values of F are not accurately defined. Of course, this problem is common to all iterative methods which directly depend on function evaluations. But, in many problems, especially in real life applications, the function values involved in the system are known only within some precision. Moreover, there are several problems which are defined by functions whose derivatives are unavailable or available at a prohibitive cost. Thus, it is very important to develop methods to face such cases.

For the second case there have been proposed several derivative free methods to contribute in this area (for example, see [2], [3]). But, for the case where the function values are known imprecisely there is a poor literature. In particular, according to our knowledge, there is one method, which is referred to as "a Newton method without evaluation of nonlinear function values", which is applied only for polynomial equations [1] and furthermore there are some other methods ideal for systems of nonlinear algebraic and/or transcendental equations, with imprecise function values [5], [6], [13].

In this paper, we propose a new Newton's method to contribute in cases of imprecise function values. In particular, we propose a new algorithm such that on one hand to remain on Newton's method because of its important quadratic convergence and on the other hand to make it suitable for problems with imprecise function values. Therefore, a new strategy is applied on Newton's method. The innovation of this strategy is the replacement of the function values $f_i(x^p)$, $i = 1, 2, \ldots, n$ in Newton's method by approximated proper quantities which are directly free of function values. Actually, this technique is equivalent to the replacement of the involved functions of the original system by other approximated, equivalent ones, to produce a new system, to apply Newton's method on it. More specifically, the key point for this transformation is the efficient utilization of proper pivot points $x_{pivot}^{p,i}$, $i = 1, 2, \ldots, n$, $p = 0, 1, \ldots$, as proposed in a previous work of ours [4]. These points correspond to the functions $f_i(x)$, $i = 1, 2, \ldots, n$ at an iteration $p = 0, 1, \ldots$. The important property of these points is that they lie on the corresponding solution surfaces of the functions $f_i(x)$ for $i = 1, 2, \ldots, n$, i.e. on the plane $x_{n+1} = 0$. Thus, at each pivot point, the corresponding function value is zero. Moreover, these points are defined lying on a line parallel to x_n axis, which passes through the current point x^p. At this point, it is worth noticing that the extraction of pivot points comes out via a componentwise technique which depends only on the signs of the functions [4], [5], [6], [12], [13].

Also, the new Newton's iterative scheme addressed in this paper, may be considered as an answer to one of three possibilities posed in [4], in order to improve Newton's method. One of these possibilities was to interfere in the function values in Newton's form-and this is just the proposed idea in this paper.

The structure of the rest of the paper is the following: In Section 2 we give the proposed method in two phases. On one hand is given the derivation of pivot points and on the other hand the main strategy of the new method. In Section 3 we give the Convergence Theorem of the new method and in Section 4 there is a geometrical interpretation of it. In Section 5 there are some numerical applications, which are implemented with the new proposed algorithm. A comparison between the new method and Newton's one is given by using problems with different properties. Numerical applications show that the proposed method is efficient and in some cases it is much better than the original Newton's one. Moreover, it works well even when a singular or an ill-conditioned Jacobian matrix is available. Lastly, in Section 6 there are some conclusions and an idea for a future work.

II. The Derivation of the proposed method

The proposed method consists of two main parts. In the first part, we extract the pivot points $x_{pivot}^{p,i}$, $i = 1, 2, \ldots, n$, $p = 0, 1, \ldots$. In the second part, utilizing properly the pivot points of part 1, we derive the proposed method from two points of view.

A. The Derivation of pivot points

We start with the derivation of pivot points $x_{pivot}^{p,i}$, $i = 1, 2, \ldots, n$, $p = 0, 1, \ldots$. As has been defined in [4], these points are appropriate points lying on the solution surfaces of the corresponding functions $f_i(x)$, $i = 1, 2, \ldots, n$. Moreover, they lie on a parallel line to x_n axis, which passes through the current point $x^p = (x_1^p, x_2^p, \ldots, x_{n-1}^p, x_n^p)$ at an iteration $p = 0, 1, \ldots$ of the algorithm.

From the definition of the pivot point, it is obvious, that these points have the same $n-1$ components with the current point x^p and differ only at the $n-$th component. Hence:

$$x_{pivot}^{p,i} = (x_1^p, \ldots, x_{n-1}^p, x_n^{p,i}), \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots. \tag{2}$$

Moreover, the corresponding functions $f_i(x)$ vanish at these points, and so

$$f_i(x_{pivot}^{p,i}) = 0, \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots. \tag{3}$$

According to the definition of *'quasi-solution property'* in [4], the relation (3) means that the pivot points have the 'quasi-solution' property.

Therefore, the unknown $n-$th component, $x_n^{p,i}$, of pivot points issues by the solution of the corresponding proper one-dimensional equations

$$f_i(x_1^p, \ldots, x_{n-1}^p, \cdot) = 0, \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots. \tag{4}$$

keeping fix the $n-1$ components $y^p = (x_1^p, \ldots, x_{n-1}^p)$.

Notation 2.1: For a better understanding of pivot points and 'quasi-solution property', see the corresponding definitions and the proper figure in [4].

Notation 2.2: The pivot points $x_{pivot}^{p,i}$, $i = 1, 2, \ldots, n$, $p = 0, 1, \ldots$ may be considered to lie on the solution surfaces of the corresponding functions $f_i(x)$, $i = 1, 2, \ldots, n$ and furthermore on a line which will pass through the current point x^p and will be parallel to x_{int}-axis, for any $int \in \{1, 2, \ldots, n\}$. In this paper, we select $int = n$.

Notation 2.3: There are several methods that can be used to solve the one-dimensional equation (4), for the production of pivot points. In this paper, we use a version of the Bisection Method, which is based on the algebraic signs of the function values [12], [13]. For this reason the proposed method, which uses these points, is a sign-function-based method. This will be more clear in the next subsection.

B. The Derivation of the proposed method

The most well known method for solving systems of nonlinear equations is Newton's method. To produce it, we obtain linear approximations to the given functions f_i at the current point x^p, $p = 0, 1, \ldots$, using the Taylor series expansion of f_i, $i = 1, 2, \ldots, n$ about this point, neglecting terms of second order and higher. So, we obtain

$$f_i(x) \approx f_i(x^p) + \sum_{j=1}^{n} \partial_j f_i(x^p)(x_j - x_j^p) \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots. \tag{5}$$

Setting equal to zero the right side of the relation (5), we get the following linear system:

$$f_i(x^p) + \sum_{j=1}^{n} \partial_j f_i(x^p)(x_j - x_j^p) = 0 \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots, \tag{6}$$

from which we end up at the general iterative form of Newton's method, given by

$$x^{p+1} = x^p - F'(x^p)^{-1} F(x^p), \quad p = 0, 1, \ldots. \tag{7}$$

Now, we are ready to produce our new method. Having in mind our goal, we target to eliminating $f_i(x^p)$ from (6). For this reason, we expand the functions f_i at the current point x^p, about the corresponding pivot points $x^{p,i}_{pivot}$, $i = 1, 2, \ldots, n$, $p = 0, 1, \ldots$.

Thus, due to (2), we have:

$$f_i(x^p) \approx f_i(x^{p,i}_{pivot}) + \sum_{j=1}^{n-1} \partial_j f_i(x^{p,i}_{pivot})(x_j^p - x_j^p) + \partial_n f_i(x^{p,i}_{pivot})(x_n^p - x_n^{p,i}), \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots. \tag{8}$$

which, due to (3), becomes

$$f_i(x^p) \approx \partial_n f_i(x^{p,i}_{pivot})(x_n^p - x_n^{p,i}), \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots \tag{9}$$

This is the crucial point. By substituting the relation (9) to (6), we obtain the following new approximated equivalent system

$$\partial_n f_i(x^{p,i}_{pivot})(x_n^p - x_n^{p,i}) + \sum_{j=1}^{n} \partial_j f_i(x^p)(x_j - x_j^p) = 0, \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots \tag{10}$$

from which we result to our new proposed iterative scheme

$$x^{p+1} = x^p - F'(x^p)^{-1} \partial_n f_i(x^{p,i}_{pivot})(x_n^p - x_n^{p,i}), \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots. \tag{11}$$

Notation 2.4: From the iterative scheme (11), it is obvious that the new proposed method is independent of function values. It depends only on the partial derivatives, $\partial_j f_i$, $j = 1, 2, \ldots, n$ of each function f_i, $i = 1, 2, \ldots, n$, evaluated at the current point x^p, $p = 0, 1, \ldots$ and the partial derivatives, $\partial_n f_i$, evaluated at the corresponding pivot points. Moreover, since the $n-$th components $x_n^{p,i}$, $i = 1, 2, \ldots$, $p = 0, 1, \ldots$ of the pivot points are unknown and they are extracted via a sign-function-based method, our new method in this sense, may be faced as a Newton's method without direct function evaluations and thus it is ideal for imprecise function problems.

Notation 2.5: From the relation (11) and the notation (2.4), it is clear that the new method costs more than Newton's one per iteration, due to the necessity of the computation of pivot points, although they issue via a low-cost technique. Thus, in cases that both methods, Newton and the proposed one, require the same number of iterations, the new method costs more than Newton's one. Nevertheless, in some cases the results are very promising. In such cases there is a sufficient reduction of the number of iterations and hence the proposed method totally costs less than Newton's one. More about them may be found in the section of Numerical Applications.

The proposed method from another point of view

Next, we will develop our new method from another point of view. From the new point of view, our method may be faced as a Newton's method on a new approximated system, equivalent to the original one. In addition, this will help us to study easier the convergence of the proposed method. In particular, we will create new appropriate linear approximations to the given functions $f_i(x)$, shot of direct function values.

Having the system (6), we transform it in an equivalent one more efficient, with the help of the pivot points. Thus, we consider the mapping

$$W = (w_1, w_2, \ldots, w_n)^T : \mathcal{D} \subset \mathcal{R}^n \to \mathcal{R}^n$$

where

$$w_i(x) = \partial_n f_i(x^{p,i}_{pivot})(x^p_n - x^{p,i}_n) + \sum_{j=1}^{n} \partial_j f_i(x^p)(x_j - x^p_j), \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots. \tag{12}$$

Hence, we have the system

$$w_i(x) = 0, \quad i = 1, 2, \ldots, n \tag{13}$$

which is equivalent to (6).

The application of Newton's method in this new posed system (13) results to the new iterative Newton's form, given by

$$x^{p+1} = x^p - W'(x^p)^{-1} W(x^p), \quad p = 0, 1, \ldots. \tag{14}$$

Next, we will prove that the new method, given by (11) is identical to (14).

For this reason, the new Newton scheme, given by (14) will be written in a more simplified form. Taking into account the relation (12) we may evaluate the function values $w_i(x^p)$. Hence, we have

$$w_i(x^p) = \partial_n f_i(x^{p,i}_{pivot})(x^p_n - x^{p,i}_n), \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots, \tag{15}$$

which are the approximations of $f_i(x^p)$ in the relation (9).

Also, we may evaluate the Jacobian matrix of W at the current point x^p. Due to (15) the functions of the new system given by (12) may now be written in the form

$$w_i(x) = w_i(x^p) + \sum_{j=1}^{n} \partial_j f_i(x^p)(x_j - x^p_j), \quad i = 1, 2, \ldots, n, \quad p = 0, 1, \ldots. \tag{16}$$

From the relation (12) or from its equivalent relation (16), it is evident that

$$W'(x) = \begin{pmatrix} \partial_1 f_1(x^p) & \partial_2 f_1(x^p) & \cdots & \partial_n f_1(x^p) \\ \partial_1 f_2(x^p) & \partial_2 f_2(x^p) & \cdots & \partial_n f_2(x^p) \\ \vdots & & & \\ \partial_1 f_n(x^p) & \partial_2 f_n(x^p) & \cdots & \partial_n f_n(x^p) \end{pmatrix} \equiv F'(x^p), \quad \forall x, \quad p = 0, 1, \ldots \tag{17}$$

and thus

$$W'(x^p) \equiv F'(x^p), \quad p = 0, 1, \ldots. \tag{18}$$

Therefore, the new iterative scheme (14) for the new posed system (13) may equivalently be given by the form

$$x^{p+1} = x^p - F'(x^p)^{-1} W(x^p), \quad p = 0, 1, \ldots. \tag{19}$$

This iterative scheme, due to (15), is equivalent to the one already referred in (11).

Notation 2.6: Although, the Newton method is applied to a new function W, there is no need to evaluate the corresponding Jacobian matrix W', which generally could be more complicated. It is based on the fact that the Jacobian of the new function W is the same with the Jacobian matrix of the original function F at the current point x^p.

III. The Convergence Theorem

In order to prove the convergence and the order of convergence of the proposed method, we will treat the method as it is posed in the above second point of view, by the relation (14). So, it is easy to implement our method as a Newton scheme on the new posed system (13). Thus the quadratic convergence of new method is evident.

With this in mind, we give the following convergence theorem for the new method.

Theorem 3.1

Suppose that $F = (f_1, \ldots, f_n) : D \subset \mathcal{R}^n \to \mathcal{R}^n$ is twice-continuously differentiable on an open neighborhood $D^* \subset D$ of a point $x^* = (x^*_1, \ldots, x^*_n) \in D$ for which $F(x^*) = 0$ and $F'(x^*)$ is nonsingular. Then the iterations $x^p, p = 0, 1, \ldots$ of the new method, given by (14) or equivalent by (19), will converge to x^* provided the initial guess x^0 is sufficiently close to x^*.

Moreover the order of convergence will be two.

Proof Using the mapping $W = (w_1, w_2, \ldots, w_n)^T : \mathcal{D} \subset \mathcal{R}^n \to \mathcal{R}^n$, given by (12), the iterates at our method are given by

$$x^{p+1} = x^p - W'^{-1}(x^p)W(x^p), \quad p = 0, 1, \ldots.$$

As we proved in the derivation of the proposed method, we have that

$$W'(x^p) = F'(x^p), \quad p = 0, 1, \ldots \tag{20}$$

where W' denotes the corresponding Jacobian of W. Hence, $W'(x^*)$ is a nonsingular matrix since by hypothesis $F'(x^*)$ is nonsingular. Thus, by the well known Newton's convergence theorem ([See [7], [9], [11]]) for our function $W(x)$ for an initial guess x^0 sufficiently close to x^*, the iterates of x^p, $p = 0, 1, \ldots$ of (14) converge to x^* and the order of convergence is two.

IV. The Geometrical Interpretation of the proposed method

A geometrical analysis of the proposed method, applied in a two-dimensional system, is shown by the following screen shot (Figure 1) looking only through the one function of them. Let the function $f_1(x_1, x_2)$ and $x^0 = (x_1^0, x_2^0)^T$ the initial point. Locate the pivot point $x_{pivot}^{0,1} = (x_1^0, x_2^{0,1})^T$. This lies on the zero-curve of function f_1, thus $f_1(x_1^0, x_2^{0,1}) = 0$. It also lies on the line $x = x_1^0$, which is parallel to the x_2- axis and passes through the point $x^0 = (x_1^0, x_2^0)^T$. The approximated quantity to the function value $f_1(x^0)$, which is given by the relation (9), is illustrated by the segment AQ.

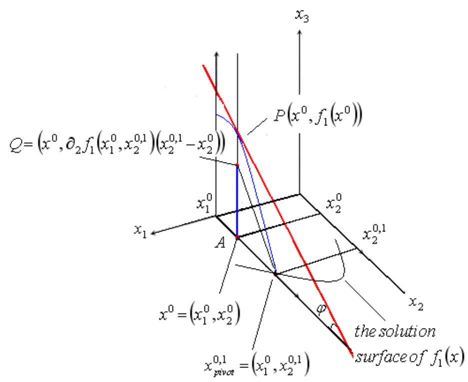

Fig. 1. A screen shot of the basic concept of the proposed method. We approximate the function value $f_1(x^0)$, denoted by the segment AP, with the function value $w_1(x^0) = \partial_2 f_1(x_{pivot}^{0,1})(x_2^0 - x_2^{0,1})$, denoted by the segment AQ.

V. Numerical Applications

Our new method for solving nonlinear systems has been applied on problems of various dimensions [6]. These problems include nonsingular, singular or ill-conditioned Jacobian matrices. The procedures have been implemented, using a FORTRAN program, named *WFEN (Without Function Evaluations Newton)* and was tested on a PC Intel Pentium.

The numerical results show the efficiency of the proposed method, especially when we have a singular or an ill-conditioned Jacobian matrix.

In Tables 1,2,3 we present the results obtained, for various initial points, by Newton's method and the proposed method with accuracy $\varepsilon = 10^{-14}$. The *'IT'* indicates the number of the iterations, *'FE'* the number of the function evaluations (including derivatives), the *'AS'* the total number of algebraic signs for the computation of the $n-$th component, $x_n^{p,i}$, of the pivot points $x_{pivot}^{p,i}$, $i = 1, 2, \ldots, n$, $p = 0, 1, \ldots$ and 'r_i' the root to which each method converges. It is worth noticing that the function evaluations for our new method are zero and we pay only for the computation of the Jacobian matrix at the current point $x^p, p = 0, 1, \ldots$ and moreover for the computation of $\partial_n f_i(x_{pivot}^{p,i})$, $i = 1, 2, \ldots, n$, $p = 0, 1, \ldots$, which are required for applying the iterative scheme (11).

Example 1:

The first system has a nonsingular Jacobian matrix. The difficulty of the system is that at some points the function values cannot be achieved accurately. It has two roots, $r_1 = (0.1, 0.1, 0.1)$ and $r_2 = (-0.1, -0.1, -0.1)$. It is given by:

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1^3 - x_1x_2x_3 = 0 \\ f_2(x_1, x_2, x_3) &= x_2^2 - x_1x_3 = 0 \\ f_3(x_1, x_2, x_3) &= 10x_1x_3 + x_2 - x_1 - 0.1 = 0 \end{aligned} \tag{21}$$

Example 2

The second system has a singular Jacobian matrix. It has the root $r = (-0.99990001*10^{-4}, -0.99990001*10^{-4}, 0.99990001*10^{-4})$. It is given by:

$$\begin{aligned} f_1(x_1, x_2, x_3) &= x_1x_3 - x_3e^{x_1^2} + 10^{-4} = 0 \\ f_2(x_1, x_2, x_3) &= x_1(x_1^2 + x_2^2) + x_2^2(x_3 - x_2) = 0 \\ f_3(x_1, x_2, x_3) &= x_1^3 + x_3^3 = 0 \end{aligned} \tag{22}$$

Example 3

The third example is the well known Brown's almost linear system. For $n = 5$, it has the three roots $r_1 = (1, 1, 1, 1, 1)$, $r_2 = (0.91635458253385, \ldots, 0.91635458253385, 1.41822708733080)$ and $r_3 = (-0.57904308849412, \ldots, -0.57904308849412, 8.89521544247060)$. In this system the Jacobian matrix is nonsingular and its difficulty is that the Jacobian matrix is ill-conditioned.

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4, x_5) &= 2x_1 + x_2 + x_3 + x_4 + x_5 - 6 = 0 \\ f_2(x_1, x_2, x_3, x_4, x_5) &= x_1 + 2x_2 + x_3 + x_4 + x_5 - 6 = 0 = 0 \\ f_3(x_1, x_2, x_3, x_4, x_5) &= x_1 + x_2 + 2x_3 + x_4 + x_5 - 6 = 0 = 0 \\ f_4(x_1, x_2, x_3, x_4, x_5) &= x_1 + x_2 + x_3 + 2x_4 + x_5 - 6 = 0 = 0 \\ f_5(x_1, x_2, x_3, x_4, x_5) &= x_1x_2x_3x_4x_5 - 1 = 0 \end{aligned} \tag{23}$$

TABLE I

COMPARISON BETWEEN WFEN AND NEWTON'S METHOD FOR EXAMPLE 1

			$Newton$			$WFEN$			
x_1^0	x_2^0	x_3^0	IT	FE	r_i	IT	FE	AS	r_i
-4	-2	1	33	396	r_2	33	396	990	r_2
-2	-0.5	0.2	32	384	r_1	32	384	960	r_1
-2	2	2	32	384	r_2	32	384	960	r_2
-1	-2	0.6	51	612	r_1	51	612	1530	r_1
-1	-2	1	29	348	r_2	29	348	870	r_2
-0.5	0.5	-0.5	26	312	r_1	26	312	780	r_1
0.4	0.5	0.5	53	636	r_2	20	240	600	r_2
0.5	-0.5	2	28	336	r_2	28	336	840	r_2
0.5	2	1	54	648	r_1	54	648	1620	r_1
2	-2	-2	43	516	r_1	43	516	1290	r_1
5	-2	-2	38	456	r_1	38	456	1140	r_1
10	-2	-2	39	468	r_1	39	468	1170	r_1

TABLE II
COMPARISON BETWEEN WFEN AND NEWTON'S METHOD FOR EXAMPLE 2

			$Newton$		$WFEN$		
x_1^0	x_2^0	x_3^0	IT	FE	IT	FE	AS
1	2	3	51	612	30	360	900
2	2	2	42	504	38	456	1140
2	2	15	387	4644	119	1428	3570
-2	-2	-2	27	324	27	324	810
3	3	3	122	1464	19	228	570
3	3	5	92	1104	43	516	1290
3	3	15	612	7344	42	504	1260
3	-2	15	612	7344	42	504	1260
4	4	4	73	876	26	312	780
-0.5	-0.5	-0.5	40	480	30	360	900
0.5	0.5	0.1	42	504	28	336	840
0.5	0.5	0.5	46	552	32	384	960

TABLE III
COMPARISON BETWEEN WFEN AND NEWTON'S METHOD FOR EXAMPLE 3

					$Newton$		$WFEN$		
x_1^0	x_2^0	x_3^0	x_4^0	x_5^0	IT	FE	IT	FE	AS
-8	-3	4	2	1.5	85	2550	8	240	400
-4	-4	4	2	1.5	80	2400	10	300	500
-2	-2	4	4	1.5	73	2190	12	360	600
-1	2	-1	2	1.5	38	1140	9	270	450
-0.5	-0.6	4	2	1.5	36	1080	9	270	450
-0.2	-0.2	-0.2	-0.2	-0.2	36	1080	34	1020	1700
-0.1	-0.1	-0.1	-0.1	-0.1	49	1470	34	1020	1700
0.1	-0.1	0.1	-0.1	4	41	1230	29	870	1450
0.1	0.1	0.1	0.1	0.1	49	1470	34	1020	1700
0.1	0.1	0.1	0.1	0.2	46	1380	29	870	1450
0.1	0.1	0.1	0.1	1	35	1050	26	780	1300
3	3	3	4	1.5	72	2160	12	360	600
10	-3	1.5	-3	1.5	81	2430	10	300	500
10	3	4	2	1.5	83	2490	12	360	600

In Table 3, the new proposed method and Newton's one converge to the root r_3 for all the initial points that we have selected, except for the initial points $(0.1, 0.1, 0.1, 0.1, 0.2)$ and $(0.1, 0.1, 0.1, 0.1, 1)$, which converge to the root r_1.

VI. CONCLUSION AND FUTURE WORK

The innovation of our work in this paper is the replacement of the original system by another appropriate and equivalent one on which we apply Newton's method. This strategy makes Newton's method very attractive in applications where function information is not available or it is computationally expensive. Thus, the new method is ideal for imprecise function problems. The basic point is the usage of the pivot points, defined in [4] which have the 'quasi-solution' property.

In a future work, we will try to improve this version, reducing the computational cost of the proposed method.

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