

Two-scale Finite Element Discretizations and Their Applications

Xingyu Gao*, Fang Liu† and Aihui Zhou‡

Abstract

This paper gives an overview for several two-scaled finite element discretization schemes for a class of elliptic partial differential equations, including both boundary value and eigenvalue problems. These schemes are based on globally and locally coupled discretizations. With these schemes, the solution of a boundary value or eigenvalue problem on a fine grid may be reduced to the solution of a boundary value or eigenvalue problem on a relatively coarse grid and the solutions of linear algebraic systems on several partially fine grids by some parallel procedure. It is shown that this type of discretization schemes not only significantly reduce the number of degrees of freedom but also produce very accurate approximations.

Key words. Combination, discretization, eigenvalue, finite element, two-scale

2000 AMS subject classifications. 65N15, 65N25, 65N30, 65N50

1 Introduction

To reduce the computational cost, including the computational time and the storage requirement, some two-scale finite element discretizations for solving partial differential equations in arbitrary dimensions are introduced in this paper. The main idea of our new discretizations is to use a coarse grid to approximate the low frequencies and to combine some univariate fine and coarse grids to handle the high frequencies by some parallel procedures. These discretizations are based on our understanding of the frequency resolution of a finite element solution to some elliptic problem. For a solution to an elliptic problem, it is shown that low frequency components can be approximated well on a relatively coarse grid and high frequency components can be computed on a fine grid (see, e.g., [5, 20, 26, 28, 33, 34, 38, 41]). It is also observed that for elliptic problems on tensor product domains, a part of high frequencies results from the tensor product of the univariate low frequencies, which can then be damped out by the tensor product of some fine and coarse grids [25, 26, 27, 28, 30].

Let us give a somewhat more detailed but informal description of the main ideas and results of the two-scale discretizations. Consider an elliptic partial differential problem in $\Omega = (0, 1)^3$. Let $P_{h_{x_1}, h_{x_2}, h_{x_3}} u$ be the standard trilinear finite element solution on a uniform grid $T^{h_{x_1}, h_{x_2}, h_{x_3}}(\Omega)$ with mesh size h_{x_1} in x_1 -direction, h_{x_2} in x_2 -direction and h_{x_3} in x_3 -direction, respectively. Then, a two-scale finite element approximation, which is nothing but a simple combination of different standard finite element solutions of the original problem over different scale meshes, is constructed as follows:

$$P_{H,H,H}^h u \equiv P_{h,H,H} u + P_{H,h,H} u + P_{H,H,h} u - 2P_{H,H,H} u,$$

where $H \gg h$. In this two-scale approximate scheme, only partially refined meshes are involved, and the following result for a class of partial differential equations can be established:

$$\|u - P_{H,H,H}^h u\|_{1,\Omega} = O(h + H^2), \quad (1.1)$$

where u is the exact solution of the partial differential equation.

*LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, P.O. Box 2719, 100080, China.

†Department of Applied Mathematics, Central University of Finance and Economics, Beijing, 100081, China.

‡LSEC, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, P.O. Box 2719, 100080, China.

This is a very satisfactory result in many ways. Consequently, for example, we obtain an asymptotically optimal approximation $P_{H,H,H}^h u$ in parallel by taking $H = O(\sqrt{h})$ and the number of degrees of freedom for obtaining $P_{H,H,H}^h u$ is only of $O(h^{-2})$, while that for the standard finite element solution $P_{h,h,h} u$ with the same approximate accuracy is of $O(h^{-3})$.

We may also design other efficient two-scale approximate schemes. For instance, consider the following eigenvalue problem posed on Ω :

$$\begin{cases} -\nabla(a\nabla u) = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where a is a positive smooth function on $\bar{\Omega}$. We may employ the following two-scale algorithm to approximate (1.2):

1. Solve (1.2) on a coarse grid: find $(\lambda_{H,H,H}, u_{H,H,H}) \in \mathbb{R} \times S_0^{H,H,H}(\Omega)$ such that $\int_{\Omega} u_{H,H,H}^2 = 1$ and

$$\int_{\Omega} a\nabla u_{H,H,H} \nabla v = \lambda_{H,H,H} \int_{\Omega} u_{H,H,H} v, \quad \forall v \in S_0^{H,H,H}(\Omega). \quad (1.3)$$

2. Compute the linear boundary value problems on partially fine grids in parallel:

find $u^{h,H,H} \in S_0^{h,H,H}(\Omega)$ such that

$$\int_{\Omega} a\nabla u^{h,H,H} \nabla v = \lambda_{H,H,H} \int_{\Omega} u_{H,H,H} v, \quad \forall v \in S_0^{h,H,H}(\Omega);$$

find $u^{H,h,H} \in S_0^{H,h,H}(\Omega)$ such that

$$\int_{\Omega} a\nabla u^{H,h,H} \nabla v = \lambda_{H,H,H} \int_{\Omega} u_{H,H,H} v, \quad \forall v \in S_0^{H,h,H}(\Omega);$$

find $u^{H,H,h} \in S_0^{H,H,h}(\Omega)$ such that

$$\int_{\Omega} a\nabla u^{H,H,h} \nabla v = \lambda_{H,H,H} \int_{\Omega} u_{H,H,H} v, \quad \forall v \in S_0^{H,H,h}(\Omega).$$

3. Set

$$u_{H,H,H}^h = u^{h,H,H} + u^{H,h,H} + u^{H,H,h} - 2u_{H,H,H}$$

and

$$\lambda_{H,H,H}^h = \frac{\int_{\Omega} a|\nabla u_{H,H,H}^h|^2}{\int_{\Omega} |u_{H,H,H}^h|^2},$$

where $S_0^{h_{x_1}, h_{x_2}, h_{x_3}}(\Omega)$ is the standard trilinear finite element space associated with $T^{h_{x_1}, h_{x_2}, h_{x_3}}(\Omega)$.

If, for example, $\lambda_{H,H,H}$ is the first eigenvalue of (1.3) at the first step, then we can establish the following results [26]

$$\left(\int_{\Omega} a|\nabla(u - u_{H,H,H}^h)|^2 \right)^{1/2} = O(h + H^2) \text{ and } |\lambda - \lambda_{H,H,H}^h| = O(h^2 + H^4).$$

These estimates mean that we can obtain asymptotically optimal approximations by taking $H = O(\sqrt{h})$. Note that what need to be solved at the second step are linear boundary value problems on partially fine grids only!

This two-scale finite element discretization method is related to the sparse grid method developed by Zenger [42], where the multi-level basis of Yserentant [40] was used. Zenger's sparse grid method is proposed for solving partial differential equations and has been known for many years in interpolation,

approximation, recovery theory and numerical quadrature under the different names "hyperbolic crosses" [1], "Boolean methods" [13], and "discrete blending" [2, 18, 22]. A general framework for approximating tensor product problems has been presented in Smolyak [32]. Zenger's sparse grid method has turned out to be a powerful approach for satisfactory numerical solutions, see, e.g., [7, 8, 16, 19, 25, 29, 30, 31]. Similar to the sparse grid method, multi-level bases are also used in wavelets (see, e.g., [7, 11, 14] and references cited therein). Instead of the multi-level basis approach, we adopt a two-level basis approach. It is shown that the two-level basis approach is more flexible than the multi-level basis approach (c.f. [15, 25, 26, 27, 28]), which is a key for us to introduce the multiscale techniques to eigenvalue problems. Moreover, since the two-scale finite element approximations are computed on regular meshes, existing solvers can be used without any need for an explicit discretization on a sparse grid.

Our approach turns out to be advantageous in two respects. First, the possibility of using existing codes allows the straightforward application of two-scale combination discretization to large scale problems. Second, since the different subproblems can be solved fully in parallel, there is a very elegant and efficient inherent coarse-grain parallelism that makes the two-scale combination discretization perfectly suitable for modern high-performance computers. One technical tool for analyzing the three-scale finite element approximation schemes is some superconvergence technique developed in [25, 26, 27, 30] (c.f., also, [23, 44]).

The remainder of this paper is organized as follows. In the coming section, some preliminary materials are provided, including some error estimations of finite element interpolants in a weak form setting. In Section 3, some two-scale finite element error estimates in [26, 27] are generalized to arbitrary dimensions (see, also, [15]). In Section 4, several numerical results, which support our theory, are reported. Finally, some concluding remarks are presented.

2 Preliminaries

Let $\Omega = (0, 1)^d$ ($d \geq 2$). We shall use the standard notation for Sobolev spaces $W^{s,p}(\Omega)$ and their associated norms and seminorms, see, e.g., [10]. For $p = 2$, we denote $H^s(\Omega) = W^{s,2}(\Omega)$ and $H_0^1(\Omega) = \{v \in H^1(\Omega) : v|_{\partial\Omega} = 0\}$, where $v|_{\partial\Omega} = 0$ is in the sense of trace, $\|\cdot\|_{s,\Omega} = \|\cdot\|_{s,2,\Omega}$ and $\|\cdot\|_\Omega = \|\cdot\|_{0,2,\Omega}$. The space $H^{-1}(\Omega)$, the dual of $H_0^1(\Omega)$, will also be used. Throughout this paper, we shall use the letter C (with or without subscripts) to denote a generic positive constant which may stand for different values at its different occurrences. For convenience, the symbol \lesssim will be used in this paper. The notation that $A \lesssim B$ means that $A \leq CB$ for some constant C that is independent of mesh parameters.

We denote by \mathbb{N}_0 the set of all nonnegative integers and $\mathbb{Z}_d = \{1, 2, \dots, d\}$. For a function $w \in W^{s,p}(\Omega)$, a point $x = (x_1, x_2, \dots, x_d) \in \Omega$ and an index $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$, we let

$$(D^\alpha w)(x) = \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_d} w}{\partial x_d^{\alpha_d}} \right) (x)$$

and

$$|\alpha| = \alpha_1 + \cdots + \alpha_d.$$

Furthermore, we denote $\mathbf{0} = (0, \dots, 0) \in \mathbb{R}^d$, $\mathbf{e} = (1, \dots, 1) \in \mathbb{R}^d$ and for $i \in \mathbb{Z}_d$, $\hat{\mathbf{e}}_i = \mathbf{e} - \mathbf{e}_i$ and $\mathbf{e}_i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^d$ whose i -th component is one and zero otherwise.

The following Sobolev space, which contains $H^3(\Omega)$, is also used (c.f. [15, 27, 28, 29, 30]):

$$W^{G,3}(\Omega) = \{w \in H^2(\Omega) : D^\alpha w \in L^2(\Omega), \mathbf{0} \leq \alpha \leq 2\mathbf{e}, |\alpha| = 3\}$$

with its natural norm $\|\cdot\|_{W^{G,3}(\Omega)}$.

2.1 A boundary value problem

Consider a homogeneous boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2.1)$$

Here L is a general linear elliptic operator of second order:

$$Lu = - \sum_{i,j=1}^d \frac{\partial}{\partial x_j} (a_{ij} \frac{\partial u}{\partial x_i}) + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} + cu$$

satisfying $a_{ij} \in W^{1,\infty}(\Omega)$, $b_i, c \in L^\infty(\Omega)$, and (a_{ij}) is uniformly positive definite on $\bar{\Omega}$.

The weak form of (2.1) is as follows: find $u \equiv L^{-1}f \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega), \quad (2.2)$$

where

$$a(u, v) = \int_{\Omega} \sum_{i,j=1}^d a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + \sum_{i=1}^d b_i \frac{\partial u}{\partial x_i} v + cuv, \quad (2.3)$$

$$(f, v) = \int_{\Omega} f v.$$

Note that for $a_0(\cdot, \cdot)$ defined by

$$a_0(u, v) = \sum_{i,j=1}^d \int_{\Omega} a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j}, \quad (2.4)$$

we have

$$\|w\|_{1,\Omega}^2 \lesssim a_0(w, w) \quad \forall w \in H_0^1(\Omega)$$

and

$$a_0(u, v) \lesssim \|u\|_{1,\Omega} \|v\|_{1,\Omega}, \quad |a(u, v) - a_0(u, v)| \lesssim \|u\|_{1,\Omega} \|v\|_{0,\Omega} \quad \forall u, v \in H_0^1(\Omega).$$

Our basic assumption is that (2.2) is well-posed, namely (2.2) is uniquely solvable for any $f \in H^{-1}(\Omega)$. (A simple sufficient condition for this assumption to be satisfied is that $c \geq 0$, see, e.g., [17, 21].) An application of the open-mapping theorem yields

$$\|w\|_{1,\Omega} \lesssim \|Lw\|_{-1,\Omega} \quad \forall w \in H_0^1(\Omega). \quad (2.5)$$

It is easy to see that if L satisfies the above assumptions and the above estimates, so does its formal adjoint

$$L^*u \equiv - \sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij} \frac{\partial u}{\partial x_j}) - \sum_{i=1}^d \frac{\partial (b_i u)}{\partial x_i} + cu.$$

We have the following estimate for the regularity of the solution of (2.1) or (2.2)

$$\|u\|_{2,\Omega} \lesssim \|f\|_{0,\Omega} \quad \forall f \in L^2(\Omega). \quad (2.6)$$

Let $T^h(\Omega)$ consist of d -rectangles, which satisfies that it is not exceedingly over-refined locally, namely, there exists $\gamma \geq 1$ such that

$$h^\gamma \lesssim h(x), \quad x \in \Omega, \quad (2.7)$$

where $h(x)$ is the mesh-size function whose value is the diameter h_τ of the element τ containing x , $h = \max_{x \in \Omega} h(x)$ is the (largest) mesh size of $T^h(\Omega)$. Define $S^h(\Omega)$ to be a space of continuous functions on Ω :

$$S^h(\Omega) = \{v \in C(\bar{\Omega}) : v|_{\tau} \in Q_1(\tau) \quad \forall \tau \in T^h(\Omega)\}, \quad (2.8)$$

where $Q_1(\tau)$ is the space of d -linear polynomials on τ when τ is an d -rectangle. Set

$$S_0^h(\Omega) = H_0^1(\Omega) \cap S^h(\Omega).$$

It is known that the standard finite element scheme for solving (2.2): Find $u_h \in S_0^h(\Omega)$ such that

$$a(u_h, v) = (f, v) \quad \forall v \in S_0^h(\Omega) \quad (2.9)$$

is well-posed if $h \ll 1$ (depending on $a(\cdot, \cdot)$, see, e.g., [33, 34]). And we can define a Galerkin-projection $P_h : H_0^1(\Omega) \mapsto S_0^h(\Omega)$ by

$$a(u - P_h u, v) = 0 \quad \forall v \in S_0^h(\Omega), \quad (2.10)$$

for which

$$\|P_h u\|_{1,\Omega} \lesssim \|u\|_{1,\Omega} \quad \forall u \in H_0^1(\Omega) \quad (2.11)$$

if $h \ll 1$.

From (2.11), various a priori global error estimates can be obtained from the approximate properties of the finite element space $S^h(\Omega)$.

2.2 An eigenvalue problem

Let $a(\cdot, \cdot) = a_0(\cdot, \cdot)$ be defined by (2.4) with regularity (2.6) for the solution of (2.2) and (a_{ij}) be symmetric. A number λ is called an eigenvalue of the form $a(\cdot, \cdot)$ relative to the form (\cdot, \cdot) if there is a nonzero vector $u \in H_0^1(\Omega)$, called an associated eigenvector, satisfying

$$a(u, v) = \lambda(u, v) \quad \forall v \in H_0^1(\Omega). \quad (2.12)$$

It is known that (2.12) has a countable sequence of real eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

and the corresponding eigenvectors

$$u_1, u_2, u_3, \dots,$$

which can be assumed to satisfy

$$(u_i, u_j) = \delta_{ij}, \quad i, j = 1, 2, \dots$$

In the sequence $\{\lambda_j\}$, the λ_j 's are repeated according to their geometric multiplicity.

A standard finite element scheme for (2.12) is: Find a pair of (λ_h, u_h) , where $\lambda_h \in \mathbb{R}$ and $0 \neq u_h \in S_0^h(\Omega)$, satisfying

$$a(u_h, v) = \lambda_h(u_h, v) \quad \forall v \in S_0^h(\Omega), \quad (2.13)$$

and use λ_h and u_h as approximations to λ and u (as $h \rightarrow 0$), respectively. One sees that (2.13) has a finite sequence of eigenvalues

$$0 < \lambda_{1,h} \leq \lambda_{2,h} \leq \dots \leq \lambda_{n_h,h}, \quad n_h = \dim S_0^h(\Omega),$$

and the corresponding eigenvectors

$$u_{1,h}, u_{2,h}, \dots, u_{n_h,h},$$

which can be assumed to satisfy

$$(u_{i,h}, u_{j,h}) = \delta_{ij}.$$

It follows directly from the minimum-maximum principle (see, e.g., [4]) that

$$\lambda_i \leq \lambda_{i,h}, \quad i = 1, 2, \dots, n_h.$$

Moreover, we have the following proposition (see [3, 4]).

Proposition 2.1 (i) *There are constants $h_0 \in (0, 1)$ and $C_i (i = 1, 2, \dots, n_h)$ such that the eigenpairs (λ_i, u_i) of (2.12) with $\|u_i\|_{0,\Omega} = 1$ and $(\lambda_{i,h}, u_{i,h})$ of (2.13) with $\|u_{i,h}\|_{0,\Omega} = 1$ can be chosen so that*

$$\|u_i - u_{i,h}\|_{1,\Omega} \leq C_i h \quad \forall h \in (0, h_0) \quad (2.14)$$

and

$$\|u_i - u_{i,h}\|_{0,\Omega} \leq C_i h \|u_i - u_{i,h}\|_{1,\Omega} \quad \forall h \in (0, h_0). \quad (2.15)$$

(ii) *For an eigenvalue,*

$$\lambda_i \leq \lambda_{i,h} \leq \lambda_i + C_i h^2. \quad (2.16)$$

The two-scale finite element analysis for eigenvalue approximations is based on the following crucial (but straightforward) property of eigenvalue and eigenvector approximation (see [4, 36]).

Proposition 2.2 *Let (λ, u) be an eigenpair of (2.12). For any $w \in H_0^1(\Omega) \setminus \{0\}$,*

$$\frac{a(w, w)}{(w, w)} - \lambda = \frac{a(w - u, w - u)}{(w, w)} - \lambda \frac{(w - u, w - u)}{(w, w)}. \quad (2.17)$$

For simplicity, we consider the approximation of any eigenvalue λ with its corresponding eigenvector u that satisfies (2.12) and $\|u\|_{0,\Omega} = 1$. Here and hereafter, we assume that (λ_h, u_h) is an associated approximation to (u, λ) that satisfy (2.14) and (2.15) with $\|u_h\|_{0,\Omega} = 1$. Consequently, we have

$$\lambda_h - \lambda + \|u - u_h\|_{0,\Omega} + h \|u - u_h\|_{1,\Omega} \lesssim h^2. \quad (2.18)$$

2.3 Finite element interpolants

To analyze the two-scale finite element approximation, we need to apply the superconvergence technique developed in [23, 26, 27, 30, 43] to arbitrary dimensions (see [15] for details).

Assume that $T^h((0, 1))$ is a uniform mesh with mesh size h on $(0, 1)$ and $S^h((0, 1)) \subset H^1((0, 1))$ is the associated piecewise linear finite element space. Set

$$S_0^h((0, 1)) = S^h((0, 1)) \cap H_0^1((0, 1)).$$

For $\mathbf{h} = (h_1, \dots, h_d)$, where $h_j \in (0, 1)$, construct a mesh of $\Omega = (0, 1)^d$ by

$$T^{\mathbf{h}}(\Omega) = T^{h_1}((0, 1)) \times \dots \times T^{h_d}((0, 1))$$

with the associated spaces of piecewise d -linear functions on Ω by

$$S^{\mathbf{h}}(\Omega) = S^{h_1}((0, 1)) \otimes \dots \otimes S^{h_d}((0, 1))$$

and

$$S_0^{\mathbf{h}}(\Omega) = S_0^{h_1}((0, 1)) \otimes \dots \otimes S_0^{h_d}((0, 1)).$$

We remark that both $S^{\mathbf{h}}(\Omega)$ and $S_0^{\mathbf{h}}(\Omega)$ are the tensor product spaces of piecewise linear functions on $(0, 1)$. The interpolation operator $I_{\mathbf{h}}$ from $C(\Omega)$ onto $S^{\mathbf{h}}(\Omega)$ is constructed by

$$I_{\mathbf{h}} = I_{h_1} \circ \dots \circ I_{h_d}.$$

For $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_d) \in \mathbb{N}_0^d$, we set

$$\mathbf{h}^\alpha = h_1^{\alpha_1} \dots h_d^{\alpha_d}$$

and

$$\mathbf{h}\alpha = (h_1\alpha_1, \dots, h_d\alpha_d).$$

The following result (see [15] for details) may be viewed as a generalization of the relevant result known in the literature (see, e.g., [23, 24, 25, 43, 44]).

Proposition 2.3 *If $w \in H_0^1(\Omega) \cap W^{G,3}(\Omega)$, then*

$$a((I - I_{\mathbf{h}})w, v) \lesssim \max_{\mathbf{0} \leq \alpha \leq \mathbf{e}, |\alpha|=2} \mathbf{h}^\alpha \|w\|_{W^{G,3}(\Omega)} \|v\|_{1,\Omega} \quad \forall v \in S_0^{\mathbf{h}}(\Omega), \quad (2.19)$$

where I is the identity operator.

3 Two-scale finite element discretizations

In this section, we will generalize the two-scale techniques in [26, 27] (c.f. [22]) to arbitrary dimensions.

Given $\sigma \in (0, 1)$. Let $w_{\mathbf{h}\alpha+\sigma\beta} \in S^{\mathbf{h}\alpha+\sigma\beta}(\Omega)$ ($\mathbf{0} \leq \alpha, \beta \leq \mathbf{e}$ and $\alpha + \beta = \mathbf{e}$), and set

$$\delta_\sigma^\alpha w_{\mathbf{h}} = \prod_{\alpha_i \neq 0} \delta_\sigma^{\mathbf{e}_i} w_{\mathbf{h}},$$

where

$$\delta_\sigma^{\mathbf{e}_i} w_{\mathbf{h}} = w_{\mathbf{h}} - w_{\mathbf{h}\hat{\mathbf{e}}_i+\sigma\mathbf{e}_i}, \quad i \in \mathbb{Z}_d.$$

If $d = 2$ and $\mathbf{h} = (h_1, h_2)$, for instance, then

$$\begin{aligned} \delta_\sigma^{(1,0)} w_{h_1, h_2} &= w_{h_1, h_2} - w_{\sigma, h_2}, \\ \delta_\sigma^{(1,1)} w_{h_1, h_2} &= w_{h_1, h_2} - w_{h_1, \sigma} - w_{\sigma, h_2} + w_{\sigma, \sigma}. \end{aligned}$$

Given $h, H \in (0, 1)$. Let $w_{H\mathbf{e}} \in S^{H\mathbf{e}}(\Omega)$, $w_{h\mathbf{e}} \in S^{h\mathbf{e}}(\Omega)$ and $w_{h\alpha+H\beta} \in S^{h\alpha+H\beta}(\Omega)$ ($\mathbf{0} \leq \alpha, \beta \leq \mathbf{e}$, $\alpha + \beta = \mathbf{e}$), and define

$$B_H^h w_{h\mathbf{e}} = w_{H\mathbf{e}} - \sum_{i=1}^d \delta_h^{\mathbf{e}_i} w_{H\mathbf{e}}.$$

It is shown in the following proposition that a one-scale interpolation on a fine grid can be obtained by some combination of two-scale interpolations asymptotically, which can be derived from the standard one-scale interpolation error estimations (c.f. [22, 26, 27, 28] for two and three dimensions).

Proposition 3.1 *There holds*

$$H \|B_H^h I_{h\mathbf{e}} w - I_{h\mathbf{e}} w\|_{1,\Omega} + \|B_H^h I_{h\mathbf{e}} w - I_{h\mathbf{e}} w\|_{0,\Omega} \lesssim H^3 \|w\|_{W^{G,3}}, \quad \text{if } w \in W^{G,3}(\Omega). \quad (3.1)$$

3.1 Two-scale finite element Galerkin projections

Recall that standard finite element scheme for solving (2.2): Find $u_{\mathbf{h}} \in S_0^{\mathbf{h}}(\Omega)$ such that

$$a(u_{\mathbf{h}}, v) = (f, v) \quad \forall v \in S_0^{\mathbf{h}}(\Omega), \quad (3.2)$$

which is well-posed when $\max\{h_i : i \in \mathbb{Z}_d\} \ll 1$ (c.f. [33, 34]). Here and hereafter, we assume that any mesh size involved is small enough so that the associated discrete problem is well-posed. For convenience, in the following discussion, $T^h(\Omega)$, $S_0^h(\Omega)$ and u_h will be replaced by $T^{\mathbf{h}}(\Omega)$, $S_0^{\mathbf{h}}(\Omega)$ and $u_{\mathbf{h}}$, respectively.

Following the two-scale finite element interpolants, we construct the two-scale finite element approximation as follows:

$$B_{H\mathbf{e}}^h u_{h\mathbf{e}} = \sum_{i=1}^d u_{H\hat{\mathbf{e}}_i+h\mathbf{e}_i} - (d-1)u_{H\mathbf{e}}.$$

For instance, $B_{H,H,H}^h u_{h,h,h} = u_{h,H,H} + u_{H,h,H} + u_{H,H,h} - 2u_{H,H,H}$.

Using Proposition 2.3, we may generalize the relevant result in [26] to arbitrary dimensions as follows [15]

Theorem 3.1 *If $u \in H_0^1(\Omega) \cap W^{G,3}(\Omega)$, then*

$$\|B_{H\mathbf{e}}^h u_{h\mathbf{e}} - u_{h\mathbf{e}}\|_{0,\Omega} + H \|B_{H\mathbf{e}}^h u_{h\mathbf{e}} - u_{h\mathbf{e}}\|_{1,\Omega} \lesssim H^3. \quad (3.3)$$

3.2 Two-scale finite element eigenvalue discretizations

In this section, we shall combine the technique in [36, 37, 38] with the multi-scale discretization approaches on tensor product grids [22, 26, 27, 28] to establish some two-scale discretizations for (2.12), where $a(\cdot, \cdot) = a_0(\cdot, \cdot)$ that is defined by (2.4) and (a_{ij}) is symmetric.

For clarity, we consider the approximation of any eigenvalue λ of (2.12) with its corresponding eigenvector u satisfying $\|u\|_{0,\Omega} = 1$. We assume that $(\lambda_{\mathbf{h}}, u_{\mathbf{h}})$ is an associated finite element approximation to (λ, u) of (2.12) in $S_0^{\mathbf{h}}(\Omega)$, namely, there holds

$$a(u_{\mathbf{h}}, v) = \lambda_{\mathbf{h}}(u_{\mathbf{h}}, v) \quad \forall v \in S_0^{\mathbf{h}}(\Omega), \quad (3.1)$$

$$\lambda_{\mathbf{h}} - \lambda + \|u_{\mathbf{h}} - u\|_{0,\Omega} + h\|u_{\mathbf{h}} - u\|_{1,\Omega} \lesssim h^2, \quad (3.2)$$

where $\|u_{\mathbf{h}}\|_{0,\Omega} = 1$ and $h = \max_{i \in \mathbb{Z}_d} \mathbf{h}^{\mathbf{e}_i} \ll 1$.

The following two-scale algorithm is proposed and analyzed in [15]:

Algorithm 3.1 1. Solve (2.12) on a globally coarse grid: find $(\lambda_{H\mathbf{e}}, u_{H\mathbf{e}}) \in S_0^{H\mathbf{e}}(\Omega) \times \mathbb{R}$ such that $\|u_{H\mathbf{e}}\|_{0,\Omega} = 1$ and

$$a(u_{H\mathbf{e}}, v) = \lambda_{H\mathbf{e}}(u_{H\mathbf{e}}, v) \quad \forall v \in S_0^{H\mathbf{e}}(\Omega).$$

2. Solve linear boundary value problems on partially fine grids in parallel: find $u_H^{\mathbf{h}\mathbf{e}_i + H\hat{\mathbf{e}}_i} \in S_0^{\mathbf{h}\mathbf{e}_i + H\hat{\mathbf{e}}_i}(\Omega)$ such that

$$a(u_H^{\mathbf{h}\mathbf{e}_i + H\hat{\mathbf{e}}_i}, v) = \lambda_{H\mathbf{e}}(u_{H\mathbf{e}}, v) \quad \forall v \in S_0^{\mathbf{h}\mathbf{e}_i + H\hat{\mathbf{e}}_i}(\Omega), \quad i \in \mathbb{Z}_d.$$

3. Set

$$u_{H\mathbf{e}}^h = \sum_{i=1}^d u^{\mathbf{h}\mathbf{e}_i + H\hat{\mathbf{e}}_i} - (d-1)u_{H\mathbf{e}},$$

$$\lambda_{H\mathbf{e}}^h = \frac{a(u_{H\mathbf{e}}^h, u_{H\mathbf{e}}^h)}{(u_{H\mathbf{e}}^h, u_{H\mathbf{e}}^h)}.$$

Theorem 3.2 Assume that $(\lambda_{H\mathbf{e}}^h, u_{H\mathbf{e}}^h)$ is the eigenpair approximation obtained by Algorithm 3.1. If $u \in H_0^1(\Omega) \cap W^{G,3}(\Omega)$, then

$$\|u - u_{H\mathbf{e}}^h\|_{1,\Omega} \lesssim H^2 + h,$$

$$|\lambda - \lambda_{H\mathbf{e}}^{k,h}| \lesssim (H^2 + h)^2.$$

It should be pointed out that the following two-scale finite element coupled approximations for eigenvalue problems are both theoretically and practically interesting in their own right. However they may be applied to devise and analyze some three-scale discretization schemes [15].

Theorem 3.3 If $u \in H_0^1(\Omega) \cap W^{G,3}(\Omega)$, then

$$|B_{H\mathbf{e}}^h \lambda_{h\mathbf{e}} - \lambda_{h\mathbf{e}}| \lesssim H^3, \quad (3.3)$$

$$\|B_{H\mathbf{e}}^h u_{h\mathbf{e}} - u_{h\mathbf{e}}\|_{1,\Omega} \lesssim H^2, \quad (3.4)$$

and

$$\|u - B_{H\mathbf{e}}^h u_{h\mathbf{e}}\|_{0,\Omega} \lesssim H^3 + h^2 \quad (3.5)$$

Note that the following estimation can be derived from Proposition 2.2 and (3.4) directly

$$\left| \frac{a(B_{H\mathbf{e}}^h u_{h\mathbf{e}}, B_{H\mathbf{e}}^h u_{h\mathbf{e}})}{(B_{H\mathbf{e}}^h u_{h\mathbf{e}}, B_{H\mathbf{e}}^h u_{h\mathbf{e}})} - \lambda \right| \lesssim (H^2 + h)^2. \quad (3.6)$$

4 Numerical examples

We have presented some two-scale finite element discretizations in Section 3. In this section, we shall present some numerical experiments that illustrate the features of our approaches [15, 26]. The numerical experiments were carried out by SGI Origin 3800 in the State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences.

We use five piecewise trilinear finite elements with the mesh sizes $h \times H \times H$, $H \times h \times H$, $H \times H \times h$, $H \times H \times H$ and $h \times h \times h$, respectively. In the first example, the two-scale finite element approximation is defined by

$$u_{H,H,H}^h = u_{h,H,H} + u_{H,h,H} + u_{H,H,h} - 2u_{H,H,H}.$$

Example 1. Consider a linear problem of three-dimensional case:

$$\begin{cases} -\sum_{i=1}^3 \frac{\partial}{\partial x_i} (x_i \frac{\partial u}{\partial x_i}) = f, & \text{in } \Omega = (1, 3) \times (1, 2) \times (1, 2), \\ u = 0, & \text{on } \partial\Omega \end{cases} \quad (4.1)$$

with the exact solution $u = (1-x)^2(3-x) \sin y(1-y)(2-y)e^z(1-z)(2-z)$.

The numerical results, presented in Tables 1 and 2, support our theory. The number of degrees of freedom for obtaining $u_{h,h,h}$ in Example 1 is about 3.4×10^7 when $h = 1/256$. It is so large that it is difficult to carry out the computation. However, it is still relatively easy to compute $u_{H,H,H}^h$ when $h = 1/256$ since the corresponding number of degrees of freedom is only 1.3×10^5 .

$2/h \times 1/H \times 1/H$	$\ u_{H,H,H}^h - u_{h,h,h}\ _1$	$\ u - u_{h,h,h}\ _1$
$8 \times 2 \times 2$	0.079664	0.231665
$32 \times 4 \times 4$	0.011148	0.057932
$128 \times 8 \times 8$	0.001428	0.014483
$162 \times 9 \times 9$	0.001005	0.011443
$200 \times 10 \times 10$	0.000733	0.009269
convergence rate	$O(H^3)$	$O(h)$

Table 1: Example 1: H^1 -estimates

$2/h \times 1/H \times 1/H$	$\ u_{H,H,H}^h - u_{h,h,h}\ _0$	$\ u - u_{h,h,h}\ _0$
$8 \times 2 \times 2$	0.005878	0.010879
$32 \times 4 \times 4$	0.000375	0.000679
$128 \times 8 \times 8$	0.000023	0.000042
$162 \times 9 \times 9$	0.000014	0.000026
$200 \times 10 \times 10$	0.000009	0.000017
convergence rate	$O(H^4)$	$O(h^2)$

Table 2: Example 1: L^2 -estimates

It is seen from Tables 1 and 2 that not only the two-scale finite element approximation $u_{H,H,H}^h$ has high accuracy but also the number of degrees of freedom for obtaining the two-scale finite element approximation $u_{H,H,H}^h$ is only of $O(1/h \times 1/H \times 1/H) = O(h^{-2})$ while that for the standard finite element solution $u_{h,h,h}$ is of $O(h^{-3})$ when $h = H^2$. For instance, the approximate accuracy of the two-scale finite element approximation $u_{H,H,H}^h$ with 2×10^4 degrees of freedom is asymptotically the same as that of the standard finite element solution $u_{h,h,h}$ with 2×10^6 degrees of freedom. Hence $u_{H,H,H}^h$ is a much better approximate solution in terms of computational cost. Moreover, the major computation can be carried out in parallel and the computational time can be reduced further.

As for solving eigenvalue problems, for illustration, we provide some numerical results obtained from Algorithm 3.1 only.

Example 2. Consider an eigenvalue problem in three-dimensions:

$$\begin{cases} -\sum_{i=1}^3 \frac{\partial}{\partial x_i} (x_i^2 \frac{\partial u}{\partial x_i}) = \lambda u, & \text{in } \Omega = (1, 3) \times (1, 2) \times (1, 2), \\ u = 0, & \text{on } \partial\Omega. \end{cases} \quad (4.2)$$

The first eigenvalue is $\lambda = \frac{3}{4} + (\frac{2}{\ln^2 2} + \frac{1}{\ln^2 2})\pi^2 \simeq 50.01212422$ and the associated eigenfunction is $u = \prod_{i=1}^3 (x_i^{-\frac{1}{2}} \sin(\frac{\pi \ln x_i}{\ln \beta_i}))$, where $\beta_1 = 3$, $\beta_2 = \beta_3 = 2$.

Here the two-scale finite element approximations are constructed as

$$u_{H,H,H}^h = u^{h,H,H} + u^{H,h,H} + u^{H,H,h} - 2u_{H,H,H}$$

and

$$\lambda_{H,H,H}^h = \frac{a(u_{H,H,H}^h, u_{H,H,H}^h)}{(u_{H,H,H}^h, u_{H,H,H}^h)}.$$

The numerical results are shown in Tables 3 and 4, which support our theory, too.

$2/h \times 1/H \times 1/H$	$\ u_{h,h,h} - u_{H,H,H}^h\ _1$	$\ u - u_{h,h,h}\ _1$
$8 \times 2 \times 2$	1.498618	1.520100
$32 \times 4 \times 4$	0.373862	0.363271
$128 \times 8 \times 8$	0.091613	0.090570
$162 \times 9 \times 9$	0.072297	0.071557
$200 \times 10 \times 10$	0.058508	0.057959
convergence rate	$O(H^2)$	$O(h)$

Table 3: Example 2: estimates for the first eigenvector

$2/h \times 1/H \times 1/H$	$ \lambda_{h,h,h} - \lambda_{H,H,H}^h $	$ \lambda - \lambda_{h,h,h} $
$8 \times 2 \times 2$	1.894330	3.363422
$32 \times 4 \times 4$	0.081407	0.203307
$128 \times 8 \times 8$	0.003940	0.012432
$162 \times 9 \times 9$	0.002414	0.007662
$200 \times 10 \times 10$	0.001568	0.004935
convergence rate	$O(H^4)$	$O(h^2)$

Table 4: Example 2: estimates for the first eigenvalue

Finally, we shall present an example in computational quantum chemistry.

Example 3. The Schrödinger equation for the Hydrogen atom is

$$-\frac{1}{2}\Delta u - \frac{1}{r}u = Eu \quad \text{in } \mathbb{R}^3, \quad (4.3)$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. The ground state energy of the Hydrogen atom is -0.5, namely, the first eigenvalue of (4.3).

We solve (4.3) in a bounded domain $\Omega = (-6.4, 6.4)^3$ and consider the homogeneous Dirichlet boundary condition: Find a pair of $(E_{\mathbf{h}}, u_{\mathbf{h}}) \in \mathbb{R} \times S_0^{\mathbf{h}}(\Omega)$ satisfying $\|u_{\mathbf{h}}\|_{0,\Omega} = 1$ and

$$\int_{\Omega} \frac{1}{2} \nabla u_{\mathbf{h}} \nabla v - \frac{1}{r} u_{\mathbf{h}} v = \int_{\Omega} u_{\mathbf{h}} v \quad \forall v \in S_0^{\mathbf{h}}(\Omega). \quad (4.4)$$

Construct the two-scale finite element approximations as follows

$$u_{H,H,H}^h = u^{h,H,H} + u^{H,h,H} + u^{H,H,h} - 2u_{H,H,H}$$

$L/h \times L/H \times L/H$	$ E - E_{H,H,H}^h $	$ E - \bar{E}_{h,h,h} $
$16 \times 8 \times 8$	0.030675	0.024474
$24 \times 12 \times 12$	0.016612	0.013946
$32 \times 16 \times 16$	0.010447	0.008965
$36 \times 18 \times 18$	0.008600	0.007417
$48 \times 24 \times 24$	0.005295	0.004591
$52 \times 26 \times 26$	0.004616	0.004004
$96 \times 24 \times 24$	0.002809	0.001377

Table 5: Example 3: estimates for the ground state energy

and

$$E_{H,H,H}^h = \frac{a(u_{H,H,H}^h, u_{H,H,H}^h)}{(u_{H,H,H}^h, u_{H,H,H}^h)}.$$

It is shown by Table 5 that the accuracy of the two-scale finite element approximations can be comparable with the standard finite element solution while the degrees of freedom reduce significantly, where $L = 12.8$.

5 Concluding remarks

In this paper, we have reviewed some two-scale finite element discretizations for elliptic boundary value problems and eigenvalue problems in arbitrary dimensions. It is shown that two-scale finite element discretizations are efficient. The main philosophy behind this paper is that to approximate multi-dimensional partial differential equations, we should use a group of finite element discretizations of different mesh scales rather than one-scale finite element discretization only [15, 26, 27, 28]. Since the computational cost and storage requirement of the multi-scale discretizations still grow exponentially with the dimensionality, however, our methods may not be applicable for very high dimensional problems.

Acknowledgements. The authors would like to thank Ms X. Dai for her help on the implementation of solving eigenvalue problems. This work was partially supported by the National Science Foundation of China under the grant 10425105 and the National Basic Research Program under the grant 2005CB321704.

References

- [1] Babenko, K.I.(1960): *Approximation by trigonometric polynomials in a certain class of periodic functions of several variables*, Dokl. Akad. Nauk SSSR, **132**, 982-985.
- [2] Baszenski, G., Delves, F.-J., and Jester, S.(1992): *Blending approximation with sine functions*, in: Numerical Methods of Approximation, Vol. 9, Braess, D. and Schumaker, L.L., eds., Int. Ser. Numer. Math., **105**, 1-19.
- [3] Babuska, I. and Osborn, J.E.(1989): *Finite element-Galerkin approximation of the eigenvalues and eigenvectors of selfadjoint problems*, Math. Comp., **52**, 275-297.
- [4] Babuska, I. and Osborn, J.E.(1991): *Eigenvalue problems*, *Handbook of Numerical Analysis*, Vol.II, Finite Element Methods(Part 1)(P.G. Ciarlet and J.L.Lions, eds.), Elsevier, 641-792.
- [5] Bank,R.E.(1996): *Hierarchical bases and the finite element method*, Acta Numerica, **5**, 1-43.
- [6] Brandt, A., McCormick, S., and Ruge, J.(1983): *Multigrid methods for differential eigenproblems*, SIAM J. Sci. Stat. Comput., **4**, 244-260.
- [7] Bungartz, H.J. and Griebel, M.(2004): *Sparse grids*, Acta Numerica, **13**, 1-123.

- [8] Bungartz, H.J., Griebel, M., and Rde, U.(1994): *Extrapolation, combination, and sparse grid techniques for elliptic boundary value problems*, Comput. Methods Appl. Mech. Engrg., **116**, 243-252.
- [9] Baszenski, G.(1991): *Remainders for Boolean interpolation*, P. Nevai and A. Pinkus. Eds. Academic Press, 31-41.
- [10] Ciarlet, P.G.(1978): *The Finite Element Method for Elliptic Problems*, North-Holland.
- [11] Cohen, A.(2003): *Numerical Analysis of Wavelets Methods*, Vol. 32, Studies in Mathematics and its Applications, North-Holland.
- [12] Costiner, S. and Ta'asan, S. (1995): *Adaptive multigrid techniques for large-scale eigenvalue problems: Solutions of the Schrdinger problem in two and three dimensions*, Phys. Rev. **E51**, 3704.
- [13] Delvos, F.J.(1982): *d-Variate Boolean interpolation*, J.Approx. Theory, **34**, 99-114.
- [14] DeVore, R., Konyagin, S.V., and Temlyakov, V.(1998): *Hyperbolic wavelet approximation*, Constr. Approx., **14**, 1-26.
- [15] Gao, X., Liu, F., and Zhou, A.(2006): *Three-scale finite element discretization schemes for eigenvalue problems*, submitted.
- [16] Garcke, J. and Griebel, M.(2000): *On the computation of the eigenproblems of hydrogen and helium in strong magnetic and electric fields with sparse grid combination technique*, J. Comput. Phys., **165**, 694-716.
- [17] Gilbarg, D. and Trudinger, N.S.(2001): *Elliptic Partial Differential Equations of Second Order*, 3rd ed., Springer-Verlag, Berlin, Heidelberg.
- [18] Gordon, W.J.(1969): *Distributive lattices and the approximation of multivariate functions*, in: Approximation with Special Emphasis on Spline Functions, Schoenberg, I.J., ed., Academic Press, New York, 223-277.
- [19] Griebel, M., Schneider, M., and Zenger, C.(1992): *A combination technique for the solution of sparse grid problem*, in: Iterative Methods in Linear Algebra, de Groen, P. and Beauwens, P., eds., IMACS, Elsevier, North Holland, 163-281.
- [20] Hackbusch, W.(1985): *Multigrid Methods and Applications*, Springer, New York.
- [21] Hackbusch, W.(1992): *Elliptic Differential Equations: Theory and Numerical Treatment*, Springer-Verlag, Berlin, Heidelberg.
- [22] Hennart, J.P. and Mund, E.H.(1988): *On the h- and p-versions of the extrapolated Gordon's projector with applications to elliptic equations*, SIAM J. Sci. Stat. Comput., **9**, 773-791.
- [23] Liao, X. and Zhou A.(1998): *A multi-parameter splitting extrapolation and a parallel algorithm for elliptic eigenvalue problems*, J. Comp. Math., **16**, 213-220.
- [24] Lin, Q. and Lin, J.(2006): *Finite Element Methods: Accuracy and Improvement*, Science Press, Beijing.
- [25] Lin, Q., Yan, N.N., and Zhou, A.(2001): *A sparse finite element method with high accuracy*, Numer. Math., **88**, 731-742.
- [26] Liu, F. and Zhou, A.(2006): *Two-scale finite element discretizations for partial differential equations*, J. Comp. Math., **24**, 373-392.
- [27] Liu, F. and Zhou, A.(2007): *Localization and parallelizations for two-scale finite element discretizations*, Comm. Pure Appl. Anal., **6**, 757-773.

- [28] Liu, F. and Zhou, A.(2007): *Two-scale Boolean Galerkin discretizations for Fredholm integral equations of the second kind*, SIAM J. Numer. Anal., **45**, 296-312.
- [29] Pflaum, C.(1997): *Convergence of the combination technique for second-order elliptic differential equations*, SIAM J. Numer. Anal., **34**, 2431-2455.
- [30] Pflaum, C. and Zhou, A.(1999): *Error analysis of the combination technique*, Numer. Math., **84**, 327-350.
- [31] Schwab, C. and Todor, R.-A.(2003): *Sparse finite elements for elliptic problems with stochastic loading*, Numer. Math., **95**, 707-734.
- [32] Smolyak, S.A.(1963): *Quadrature and interpolation formulas for tensor products of certain classes of functions*, Dokl. Akad. Nauk SSSR., **148**, 1042-1045.
- [33] Xu, J.(1996): *Two-grid discretization techniques for linear and nonlinear PDEs*, SIAM J. Numer. Anal., **33**, 1759-1777.
- [34] Xu, J. and Zhou, A.(2000): *Local and parallel finite element algorithms based on two-grid discretizations*, Math. Comp., **69**, 881-909.
- [35] Xu, J. and Zhou, A.(2001): *Local and parallel finite element algorithms based on two-grid discretizations for nonlinear problems*, Adv. Comp. Math., **14**, 293-327.
- [36] Xu, J. and Zhou, A.(2001): *A two-grid discretization scheme for eigenvalue problems*, Math. Comp., **70**, 17-25.
- [37] Xu, J. and Zhou, A. (2002): *Local and parallel finite element algorithms for eigenvalue problems*, Acta Math. Appl. Sinica, English Series, **18**, 185-200.
- [38] Xu, J. and Zhou, A.(2002) *Some multiscale methods for parial differential equations*, Contemp. Math., **306**, 1-27.
- [39] Xu, Y. and Zhou, A.(2004): *Fast Boolean approximations methods for solving integral equations in high dimensions*, J. Integral Equ. Appl, **16**, 83-110.
- [40] Yserentant, H. (1986): *On the multi-level splitting of finite element spaces*, Numer. Math., **49**, 379-412.
- [41] Yserentant, H.(1993): *Old and new proofs for multigrid algorithms*, Acta Numerica, **2**, 285-326.
- [42] Zenger, C.(1991): *Sparse grids*, in: Parallel Algorithm for Partial Differential Equations, Proc. of 6th GAMM-Seminar, Hackbush, W., ed., Braunschweig, Vieweg-Verlag.
- [43] Zhou, A. and Li, J.(1994): *The full approximation accuracy for the stream function-vorticity-pressure method*, Numer. Math., **68**, 401-411.
- [44] Zhou, A., Liem, C., Shih, T., and Lu, T.(1997): *A multi-parameter splitting extrapolation and a parallel algorithm*, Sys. Sci. Math. Sci., **10**, 253-260.
- [45] Zhou, A., Liem, C., Shih, T., and Lu, T. (1998): *Error analysis on bi-parameter finite elements*, Comput. Methods Appl. Mech. Engrg., **158**, 329-339.