

Generating formal power series and stability of bilinear systems

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Abstract

The aim of this paper is to study the Bounded Input Bounded Output (BIBO) stability of bilinear systems. The stability of linear systems can be studied by computing their transfer function. In this paper, we use the generating series (generalization of the transfer function) as a tool for analysing the stability of bilinear systems. In fact, the generating series G of a bilinear system is a formal power rational series in noncommutative variables. It provides a formal expression of the output $y = \varepsilon(G)$ in iterated integrals form.

The stability/stabilization can always be studied from the generating series G : According to expression of G , three cases occur. In the first case, the output $y = \varepsilon(G)$ can be explicitly computed; in the second case, this output can be bounded/unbounded if the input $u(t)$ is bounded; in the third case, no conclusion about the BIBO stability can be easily deduced. We look only for a stabilizing constant input $u(t) = \eta$, by studying the univariate series G_η .

Index Terms

BIBO stability, Bilinear systems, Generating series in noncommutative variables.

I. INTRODUCTION

The stability and the stabilization of dynamical systems have been studied by many authors [7], [8]. The BIBO (Bounded Input Bounded Output) stability is especially interesting for the dynamical systems describing some biological phenomena (for instance, the modeling of the behavior insulin/glycaemia of diabetics where biological parameters have to be bounded). In previous papers, we proposed a modeling of affine dynamical system by bilinear ones [1], [9], particularly in the field of diabetes [5], [6]. The objective of this article is to study the stability of bilinear systems for a single input with drift. This problem has already been studied by approximate methods [2], [11] or by noting the perturbations [12]. Our method consists in using the generating series (in noncommutative variables) computed at the beginning of the modeling of the affine dynamical system, as a rational expression.

II. PRELIMINARIES

We consider unknown systems assumed to be affine dynamical systems of the form

$$(\Sigma) \begin{cases} \dot{x}(t) &= g_0(x) + \sum_{i=1}^m u_i(t)g_i(x) \\ y &= h(x) \end{cases} \quad (1)$$

with

- state vector $x \in \mathcal{V}$, analytical variety, vector fields $g_i = \sum_{s=1}^N g_i^s \frac{\partial}{\partial x^s} \quad \forall i \quad 0 \leq i \leq m$,
- inputs $\{u_i\}$ piecewise continuous, observation function $h : \mathcal{V} \rightarrow \mathbb{R}$.

We approximate (Σ) by a bilinear system, according to the following method:

For any order k , we compute the generating series G_k of (Σ) up to order k and we construct a bilinear system (B_k) the generating series of which is G_k . Then the Taylor expansions of the outputs $y(t)$ of (Σ) and $y_k(t)$ of B_k coincide up to order k . So this approximation of (Σ) by a bilinear system (B_k) is better than an approximation by any linear system (L) (from order 3, the Taylor expansions of the associated outputs becoming generally different (see III.A)).

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A. Bilinear system

Let B be the bilinear system given by its state equations:

$$(B) \begin{cases} \dot{x}(t) &= (M_0 + \sum_{i=1}^m u_i(t)M_i)x(t) \\ y(t) &= \lambda.x(t) \end{cases} \quad (2)$$

- $x(t)$ belongs to an \mathbb{R} -vector space Q , $\{M_i\}_{0 \leq i \leq m} : Q \rightarrow Q$ and $\lambda : Q \rightarrow \mathbb{R}$ are \mathbb{R} -linear.
- inputs $\{u_i\}_{0 \leq i \leq m}$ are piecewise continuous, ($u_0(t) \equiv 1$ for the drift)

The generating series G [3] of (B) is rational, defined on the alphabet $Z = \{z_i\}_{0 \leq i \leq m}$ corresponding to $\{u_i\}_{0 \leq i \leq m}$:

$$G = \lambda.x(0) + \sum_{\nu \geq 0} \sum_{j_0, \dots, j_\nu = 0}^m \lambda.M_{j_0} \cdots M_{j_\nu} x(0) z_{j_0} \cdots z_{j_\nu}$$

B. Bounded Input Bounded Output Stability: BIBO stability

A dynamical system is Bounded Input Bounded Output stable if and only if its output $y(t)$ is defined for every bounded input $u(t)$ and if for every bounded input, the output is bounded.

III. BILINEAR SYSTEM WITH A BOUNDED EXCITATION

We suppose that the dynamical system has a single input with drift:

$$(B) \begin{cases} \dot{x}(t) &= (M_0 + u_1(t)M_1)x(t) \\ y(t) &= \lambda.x(t) \end{cases} \quad (3)$$

The output $y(t)$ is given by the Fliess equation: $y(t) = \sum_{w \in Z^*} \langle G|w \rangle \int_0^t \delta(w)$ where G is the generating series of (B) and $\int_0^t \delta(w)$ is the iterated integral of $w \in Z^* = \{z_0, z_1\}^*$. Let us remind

$$\begin{cases} \int_0^t \delta(\epsilon) &= 1 \\ \int_0^t \delta(vz_i) &= \int_0^t \left(\int_0^\tau \delta(v) \right) u_i(\tau) d\tau \\ &\forall z_j \in Z \quad \forall v \in Z^*. \end{cases}$$

A. Computation and study of the generating series of a bilinear system (B)

If we only know the state equations (B) , we can compute its generating series as follows.

1) *Computation of the generating series of a bilinear system:* :

- *Method*[9] generalizing the Schutzenberger's method [13] for computing the rational expression describing a rational series

- 1) From the representation $(M_0, M_1, \lambda, \gamma)$ of the system (B) , the dimension of which is n , we build a finite weighed automaton $\langle Z, Q, \gamma, A, \delta \rangle$ where
 - $Q = \{q_1, \dots, q_n\}$ is the set of states, $Z = \{z_0, z_1\}$ is the alphabet, δ is the transition map
 - γ is the initial state, A is the set of the final states associated with λ
- 2) From the weighed automaton, we build a rational expression We write the system of equations satisfied by the automaton at every state $q_j \in Q$.

$$R_j = \lambda_j^{(0)} + \sum_{i=1}^n A_{ji}^{(0)} R_i, \quad 1 \leq j \leq n, \quad \lambda_j^{(0)} \in \mathbb{R}$$

where $A_{ji}^{(0)} = a_{ji}z_0 + b_{ji}z_1$, $a_{ji}, b_{ji} \in \mathbb{R}$. By eliminations, we obtain the rational expression at q_1 : $R_1 = \lambda_1^{(n-1)} + A_{11}^{(n-1)} R_1$. More precisely, we use the following rule (R) :

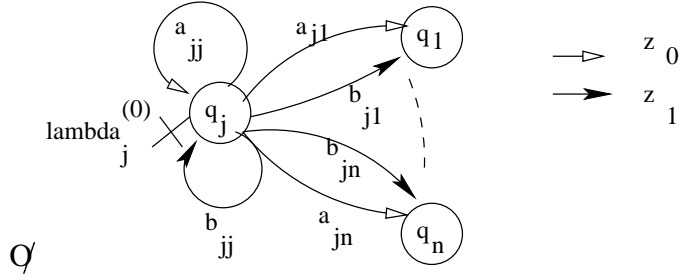


Fig. 1. weighed automaton cell at state q_j

“If $R_j = \Delta R_j + \Gamma$ for some regular expressions Δ, Γ and for Δ proper, then $R_j = \Delta^* \Gamma$ ”
 And $\forall p, n \geq p \geq 2$, we substitute this expression into R_p in order to reduce the size of the system from p to $p - 1$ equations. Then we obtain, for $1 \leq k, i \leq p - 1$:

$$\begin{cases} A_{ki}^{(n-p+1)} &= A_{ki}^{(n-p)} + A_{kp}^{(n-p)} A_{pp}^{(n-p)*} A_{pi}^{(n-p)} \\ \lambda_k^{(n-p+1)} &= \lambda_k^{(n-p)} + A_{kp}^{(n-p)} A_{pp}^{(n-p)*} \lambda_{k+1}^{(n-p)} \end{cases}$$

Then the rational generating series associated with (B) is $G = R_1 = A_{11}^{(n-1)*} \lambda_1^{(n-1)}$

- *Example : bilinear system approximating the electric equation at order 2*

The nonlinear differential “electric equation” [4] is given by:

$$\dot{v}(t) = -k_1 v(t) - k_2 v^2(t) + u(t) \quad (4)$$

By setting

$$x(t) = v(t), \quad a^{(0)} = -k_1 x(0) - k_2 x(0)^2, \quad a^{(1)} = -k_1 - 2k_2 x(0), \quad a^{(2)} = -2k_2$$

$$A_0 = a^{(0)} \frac{d}{dx}, \quad A_1 = \frac{d}{dx}$$

the electric equation can be written as an affine dynamical system

$$\dot{x}(t) = A_0(x) + A_1(x)u(t), \quad y(t) = x(t) \quad (5)$$

Its generating series G is not rational. It can be written as a continued fraction (Fig.2).

$$G = \frac{(z_1 + a^{(0)} z_0) | (a^{(1)} z_0)^*}{1 - C_1} + x(0)$$

denoting $(z_1 + a^{(0)} z_0) C_1^* (a^{(1)} z_0)^* + x(0)$, with

$$C_n = \frac{(a^{(1)} z_0)^* (z_1 + a^{(0)} z_0) | \left(\binom{n}{1} a^{(1)} z_0 \right)^* \left(\binom{n}{2} a^{(2)} z_0 \right)}{1 - C_{n+1}}, \quad n \geq 1$$

A bilinear system (B_2) approximating it at order 2 is, by truncating the automaton of G

$$(B_2) \begin{cases} \dot{x}(t) &= \left(\begin{pmatrix} 0 & 0 \\ a^{(0)} & a^{(1)} \end{pmatrix} + u(t) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right) x(t) \\ y(t) &= (x(0) \ 1) x(t) \end{cases}$$

The rational expression associated with this automaton is:

$$G_2 = (a^{(0)} z_0 + z_1) (a^{(1)} z_0)^* + x(0)$$

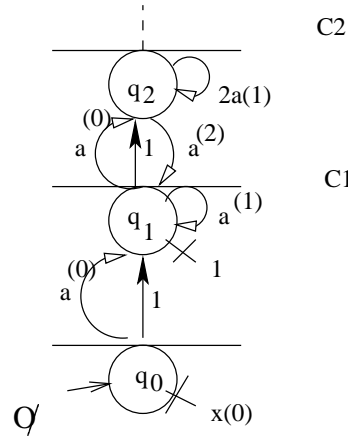


Fig. 2. electric series automaton

2) *Remark: Computation of the generating series of a linear system* : Let a linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = \lambda x(t) \end{cases}$$

where A, B, λ are matricial. Its transfer function is $H(s) = \lambda(s.Id - A)^{-1}B$, its generating series is

$$G = \sum_{n=0}^{\infty} \lambda A^n x(0) z_0^n + \sum_{n=0}^{\infty} \lambda A^n B z_1 z_0^n = \lambda z_1 (Id - z_0 A)^{-1} B + \lambda (Id - z_0 A)^{-1} x(0)$$

The poles of G are the inverses of the eigenvalues of A . And then, the stability can be studied by computing the generating series $G = z_1 \frac{1}{z_0} H(\frac{1}{z_0}) + \lambda (Id - z_0 A)^{-1} x(0)$. And G only contains words z_0^i and $z_1 z_0^j$ and not more general words.

B. Study of the stability of a system according to its generating series

The output $y(t)$ is obtained by “evaluating its generating series G ” [3]:

$$y(t) = \varepsilon(G) = \sum_{w \in Z^*} \langle G|w \rangle \int_0^t \delta(w)$$

We split the problem of the BIBO stability in several cases according to the form of G .

1) • *First case: the output $y = \varepsilon(G)$ can be explicitly computed* :

- G is a simple rational expression : $G = z_i^p (c_j z_j)^*$, $(c_i z_i)^* z_j^p$, $i, j = 0..1, \dots$, $c_i, c_j \in \mathbb{R}$

We use some results about the evaluation of a series [10] in the following table.

(for $\xi_{z_1}(t) = \int_0^t u(\tau) d\tau$ with $\xi_{z_1}(0) = 0$ and $\xi_{z_0}(t) = \int_0^t d\tau = t$.)

We study the BIBO stability according to the form of G .

- $G = (c_0 z_0)^*$ for $c_0 \neq 0$: G represents a (bi)linear system of dim 1, without excitation.

$$y(t) = \sum_{i=0}^{\infty} c_0^i \frac{t^i}{i!} = e^{c_0 t}$$

This output is bounded iff $c_0 < 0$. Then the system is BIBO iff $c_0 < 0$.

- $G = (c_1 z_1)^*$ for $c_1 \neq 0$: G represents a bilinear system of dim 1 without drift.

$$y(t) = \sum_{i=0}^{\infty} \frac{c_1^i \xi(t)^i}{i!} = e^{c_1 \xi(t)}$$

And then the system is not BIBO:

- * if $c_1 > 0$ then for every input $u(t)$, the primitive of which is $\xi(t) > 0$, the output is unbounded.
 - * if $c_1 < 0$ then for every input $u(t)$, the primitive of which is $\xi(t) < 0$, the output is unbounded.
- For every input $u(t) > 0$, then $\xi(t) > 0$ and the system is PBIBO (positive bounded input bounded output). Then the system is PBIBO iff $c_1 < 0$

Series G	Iterated integral $\int_0^t \delta(G) = \varepsilon(G)$
ϵ	$\int_0^t \delta(\epsilon) = 1$
z^n	$\int_0^t \delta(z^n) = \frac{\xi_z^n(t)}{n!}$
$\sum_{n \geq 0} z^n$	$\int_0^t \delta(\sum_{n \geq 0} z^n) = \exp(\xi_z(t))$
$(z^*)^n$	$\int_0^t \delta((z^*)^n) = \exp(\xi_z(t)) \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(\xi_z(t))^j}{j!}$
$\sum_{n \geq 0} (-1)^n z^{2n}$	$\int_0^t \sum_{n \geq 0} (-1)^n z^{2n} = \cos(\xi_z(t))$
$\sum_{n \geq 0} (-1)^n z^{2n+1}$	$\int_0^t \sum_{n \geq 0} (-1)^n z^{2n+1} = \sin(\xi_z(t))$

- $G = z_0^p(c_0 z_0)^*$, $z_0^p(c_1 z_1)^*$, $z_1^p(c_0 z_0)^*$, $z_1^p(c_1 z_1)^*$, $(c_0 z_0)^* z_1^p$, $(c_1 z_1)^* z_0^p, \dots$

Let us consider these different cases:

- 1) $G = z_0^p(c_0 z_0)^*$ for $p \geq 1$, $c_0 \neq 0$

G represents a bilinear system of dim $p + 1$ without excitation.

$$y(t) = \frac{1}{c_0^p} (e^{c_0 t} - \sum_{i=0}^{p-1} c_0^i \frac{t^i}{i!})$$

This system is not BIBO if $c_0 > 0$, not BIBO if $c_0 < 0$ and $p > 1$, is BIBO iff $c_0 < 0$, $p = 1$

- 2) $G = z_0^p(c_1 z_1)^*$ for $p > 0$, $c_1 \neq 0$

G represents a bilinear single input system of dim $p + 1$ with drift.

For a constant input $u(t) = \eta$, the output is

$$y(t) = \sum_{i=0}^{\infty} c_1^i \eta^i \frac{t^{p+i}}{(p+i)!} = \frac{1}{c_1^p \eta^p} (e^{c_1 \eta t} - \sum_{i=0}^{p-1} c_1^i \eta^i \frac{t^i}{i!})$$

For $c_1 > 0$, the system is not BIBO (for $\eta > 0$), for $c_1 < 0$, the system is not BIBO (for $\eta < 0$). For $(c_1 < 0, p = 1)$: $u(t) = \eta > 0$ is a stabilizant input.

- 3) $G = z_1^p(c_0 z_0)^*$ for $c_0 \neq 0$

G represents a bilinear single input system of dim $p + 1$ with drift.

For a constant input $u(t) = \eta$, the output is

$$y(t) = \sum_{i=0}^{\infty} c_0^i \eta^p \frac{t^{p+i}}{(p+i)!} = \frac{\eta^p}{c_0^p} (e^{c_0 t} - \sum_{i=0}^{p-1} c_0^i \frac{t^i}{i!})$$

For $c_0 > 0$, or for $(c_0 < 0, p > 1)$, the system is neither BIBO (nor PBIBO).

For $(c_0 < 0, p = 1)$: $u(t) = \eta > 0$ is a stabilizant input.

- 4) $G = z_1^p (c_1 z_1)^*$ for $c_1 \neq 0$

For a constant input $u(t) = \eta$, the output is

$$y(t) = \sum_{i=0}^{\infty} c_1^i \eta^{(p+i)} \frac{t^{p+i}}{(p+i)!} = \frac{1}{c_1^p} (e^{c_1 \eta t} - \sum_{i=0}^{p-1} c_1^i \eta^i \frac{t^i}{i!})$$

For $c_1 > 0$, the system is not BIBO (for $\eta > 0$), for $c_1 < 0$, the system is not BIBO (for $\eta < 0$). For $(c_1 < 0, p = 1) : u(t) = \eta > 0$ is a stabilizant input.

- 5) $G = (c_0 z_0)^* z_1^p$ for $c_0 \neq 0$

For a constant input $u(t) = \eta$, the output is

$$y(t) = \sum_{i=0}^{\infty} c_0^i \eta^p \frac{t^{p+i}}{(p+i)!} = \frac{\eta^p}{c_0^p} (e^{c_0 t} - \sum_{i=0}^{p-1} c_0^i \frac{t^i}{i!})$$

For $c_0 > 0$ or for $(c_0 < 0, p > 1)$, the system is neither BIBO (nor PBIBO).

For $(c_0 < 0, p = 1) : u(t) = \eta > 0$ is a stabilizant input.

- 6) $G = (c_1 z_1)^* z_0^p$ for $c_1 \neq 0$

For a constant input $u(t) = \eta$, the output is

$$y(t) = \sum_{i=0}^{\infty} c_1^i \eta^i \frac{t^{p+i}}{(p+i)!} = \frac{1}{c_1^p \eta^p} (e^{c_1 \eta t} - \sum_{i=0}^{p-1} c_1^i \eta^i \frac{t^i}{i!})$$

For $c_1 > 0$, the system is not BIBO (for $\eta > 0$), for $c_1 < 0$, the system is not BIBO (for $\eta < 0$). For $(c_1 < 0, p = 1) : u(t) = \eta > 0$ is a stabilizant input.

- G is a shuffle product of several generating series $G_1 \cdots G_p$: $G = G_1 \sqcup G_2 \sqcup \cdots \sqcup G_p$

We use the theorem proved by M.Fliess [3]

Theorem 1:

G_1 and G_2 being the generating series of some affine dynamical systems $(\Sigma_1), (\Sigma_2)$, then the system the output of which is the product of the outputs of (Σ_1) and (Σ_2) , has the generating series $G_1 \sqcup G_2$.

In other words, $\varepsilon(G_1 \sqcup G_2) = \varepsilon(G_1)\varepsilon(G_2)$.

For instance, if $G = (c_0 z_0 + c_1 z_1)^*$, then $G = (c_0 z_0)^* \sqcup (c_1 z_1)^*$ and $y(t) = e^{c_0 t} e^{c_1 \xi(t)}$

- G is an exchangeable series (the coefficient of every word is independent of the order of the letters in the word), G can be written $G = \sum_{i_0, i_1} g_{i_0 i_1} z_0^{i_0} \sqcup z_1^{i_1}$ and $y(t) = \sum_{i_0, i_1} g_{i_0 i_1} \varepsilon(z_0^{i_0}) \varepsilon(z_1^{i_1})$

$$y(t) = \sum_{i_0, i_1} g_{i_0 i_1} \frac{t^{i_0} \xi^{i_1}}{i_0! i_1!}$$

2) • *Second case: the output is bounded/ unbounded if the input $u(t)$ is bounded. : G is obtained by concatenating some simple rational expressions*

$$(c_0 z_{j_0})^{*p_0} z_{i_1} (c_1 z_{j_1})^{*p_1} \cdots z_{i_k} (c_k z_{j_k})^{*p_k} \quad (F)$$

We use the theorem proved by Hoang Ngoc Minh [10] :

Theorem 2:

For every positive integer k , let us suppose that G_k is an exchangeable series and let us denote by $g_k(\xi(t))$ its evaluation $g_k(\xi(t)) = g_k(t, \xi_1(t), \cdots, \xi_m(t))$ where $\xi_j(t)$ is the primitive of the input $u_j(t)$ cancelling for $t = 0$. Then, for every positive integer k , the series $S_k = G_0 z_{i_1} G_1 \cdots z_{i_k} G_k$ where z_{i_1}, \cdots, z_{i_k} are some letters of Z , has the following evaluation:

$$y(t) = \int_0^t \int_0^{\tau_k} \cdots \int_0^{\tau_2} g_0(\xi(\tau_1)) g_1(\xi(\tau_2) - \xi(\tau_1)) \cdots g_k(\xi(t) - \xi(\tau_k)) d\xi_{z_{i_1}}(\tau_1) \cdots d\xi_{z_{i_k}}(\tau_k)$$

Remark : When the BIBO stability is proved, the output can be bounded for G of the form (F) .

example : $G = (c_1 z_0)^* z_1 (c_2 z_0)^* z_1 (c_3 z_0)^*$

$$y(t) = \int_0^t \left(\int_0^{\tau_2} e^{c_1 \tau_1} e^{c_2 (\tau_2 - \tau_1)} e^{c_3 (t - \tau_2)} u(\tau_1) d\tau_1 \right) u(\tau_2) d\tau_2$$

For $0 \leq u(t) \leq M$, $c_1 < 0$, $c_2 < 0$, $c_3 < 0$, then

$$y(t) \leq \frac{M^2 e^{\max(c_1, c_2, c_3)t}}{(\max(c_1, c_2) - \min(c_1, c_2))(\max(c_3, \max(c_1, c_2)) - \min(c_3, \max(c_1, c_2)))}$$

3) • *Third case: no conclusion seems available about the BIBO stability by using G* : We look only for a stabilizing constant input $u(t) = \eta$, by studying the univariate series G_η .

We prove the following proposition and corollaries:

Proposition 1

The output $(\varepsilon(G))_\eta$ of a single input with drift system, the generating series of which is G , for the input $u(t) = \eta$, is equal to the output $\varepsilon(G_\eta)$ of some system, the generating series of which is G_η , obtained by substituting ηz_0 to z_1 in G .

Proof

$$(\varepsilon(G))_\eta = \left(\sum_{w \in \{z_0, z_1\}^*} \langle G|w \rangle \int_0^t \delta(w) \right)_\eta$$

For $u(t) = \eta$,

$$\int_0^t \delta(z_{i_1} \cdots z_{i_l}) = \eta^{|z_{i_1} \cdots z_{i_l}|_{z_1}} \int_0^t \delta(z_0^l) = \eta^{|z_{i_1} \cdots z_{i_l}|_{z_1}} \frac{t^l}{l!}$$

$$(\varepsilon(G))_\eta = \sum_{l \geq 0} \sum_{k=0}^l \sum_{|w|_{z_1}=k, |w|=l} \langle G|w \rangle \eta^k \frac{t^l}{l!}$$

$$G_\eta = \sum_{l \geq 0} \sum_{k=0}^l \sum_{|w|_{z_1}=k, |w|=l} \langle G|w \rangle z_0^l \eta^k, \quad (\varepsilon(G))_\eta = \varepsilon(G_\eta)$$

Corollary 1

A necessary condition for BIBO stability of bilinear system, the generating series of which is G , is that, $\forall \eta \in \mathbb{R}$, the real part of the poles of G_η is ≤ 0 and the imaginary poles of G_η are single.

Proof

Suppose that for every bounded input $u(t)$, the associated input $y(t)$ is bounded. G being rational, then G_η is. G_η being a single variable rational series, the BIBO stability can be studied by decomposing it in partial fractions :

$$G_\eta = P(z_0) + \sum_{e_i=1}^{m_i} \frac{\alpha_i}{(1 - a_i z_0)^{e_i}}, \quad P(z_0) \in C[z_0]$$

And if $\deg(P(z_0)) = 0$, $\varepsilon(G_\eta) = \sum_{e_i=1}^{m_i} \alpha_i \exp(a_i t) \sum_{j=0}^{e_i-1} \binom{e_i-1}{j} \frac{a_i^j t^j}{j!}$

And necessarily, $\forall i$, $\text{Real}(a_i) \leq 0$ and $(\text{Real}(a_i) = 0) \Rightarrow (e_i = 1)$

Corollary 2

If there exists η such that every pole of G_η has a negative real part and if every imaginary pole is single, then $u(t) = \eta$ is a stabilizing input

Example : Bilinear approximants of the electric equation

1) *At order 2*

The generating series is, for $a^{(1)} \neq 0$:

$$G_2 = (z_1 + a^{(0)}z_0)(a^{(1)}z_0)^* + x(0)$$

By using the theorem 2, $G_{21} = z_1(a^{(1)}z_0)^*$, $G_{22} = a^{(0)}z_0(a^{(1)}z_0)^*$, with $G_2 = G_{21} + G_{22} + x(0)$, and $\varepsilon(G_{21}) = e^{a^{(1)}t} \int_0^t e^{-a^{(1)}\tau_1} d\xi_1(\tau_1)$ $\varepsilon(G_{22}) = a^{(0)}e^{a^{(1)}t} \int_0^t e^{-a^{(1)}\tau_1} d\tau_1$

This system is not BIBO for $a^{(1)} > 0$ and is BIBO for $a^{(1)} < 0$

For instance, for $x(0) \geq 0$, $a^{(0)} > 0$, $a^{(1)} < 0$, $0 \leq u(t) \leq M$, then $y(t) \leq x(0) + \frac{M+a^{(0)}}{-a^{(1)}}$

2) *At order 3*

The generating series is, for $a^{(1)} \neq 0$, $a^{(2)} \neq 0$:

$$G_3 = (z_1 + a^{(0)}z_0)(a^{(1)}z_0 + (z_1 + a^{(0)}z_0)a^{(2)}z_0)^* + x(0)$$

We compute $G_{3,\eta}$ by substituting ηz_0 to z_1 in G_3 : $G_{3,\eta} = x(0) + \frac{(a^{(0)}+\eta)z_0}{1-a^{(1)}z_0-(a^{(0)}+\eta)a^{(2)}z_0^2}$. If $\eta = -a^{(0)}$ then $y_{3,\eta}(t) = x(0)$ else we decompose $G_{3,\eta}$ in partial fractions for studying the stabilizing inputs.

3) *At order $k > 3$*

$$G_\eta = \frac{((\eta + a^{(0)})z_0)|(a^{(1)}z_0)^*}{1 - C_1} + x(0)$$

with

$$C_n = \frac{(a^{(1)}z_0)^*((\eta + a^{(0)})z_0)|\left(\binom{n}{1}a^{(1)}z_0\right)^*\left(\binom{n}{2}a^{(2)}z_0\right)}{1 - C_{n+1}}, \quad n \geq 1$$

And we study the poles of $G_{k,\eta}$. For $\eta = -a^{(0)}$, $G_{k,\eta} = x(0)$, $\forall k$ and the system is stationary.

IV. CONCLUSION

The BIBO stability of bilinear systems cannot be generally studied by using their state equation. The method, proposed in this paper, consists in using the ‘‘evaluation’’ of their generating series G . If the generating series is in the following forms - simple rational, obtained by shuffle product, obtained by concatenating some simple rational expressions -, then the use of the generating series provides an answer about the stability and a bound for the output. According to M.Fliess [4], the generating series of a lot of dynamical systems (modeling of nonlinear electronic circuits, for instance), can be approximated by a linear combination of rational expressions of the previous forms. Otherwise, we can look for a stabilizant input $u(t) = \eta$, by using the generating univariate series.

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