Validation of a particular class of dynamical systems

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Abstract

The aim of this paper is to study the validation of a model for non-linear systems. It evaluates the ability of a model to simulate the behavior of an unknown system (Σ). This is a central problem in identification ([2]).
In almost cases, validation consists, by a statistical approach, in a test that falsifies or not falsifies the model, using a given discrete sampled data set.
Our approach, based on combinatorial techniques, is different and provides :

- an exact symbolic computation of the error E, due to the approximation of (Σ) by a family of bilinear systems described by generating series. This produces an estimation of E, that is essential in the measure of model's quality [2]. Particularly, we can determine intervals where our model is "acceptable", independently of the data set choosen for validation, as in discret case.
- a computing tool parameterized by the input and the system's behavior. In fact, our computation is a sum of differential monomials in the input functions and behavior system. So, we can determine and separate input and system contributions to this error and we obtain, in this way, a valuation process for rough and oscillating inputs as well as smooth inputs.

1 Introduction

The model validation is a crucial problem in system identification[2]. It measures confidence in the model to reproduce the behavior of a dynamic system, under some hypothesis.
In a discret-time approach, the model validation is really an invalidation since it determines wether a discret sample input-output is inconsistent with the model.
But, the question of experiment design for model invalidation arises [3]. Jung, in this paper, points out this problem of accepting the simplest possible model that has not been falsified by experimental data and therefore putting aside some important properties of the real system. To avoid this, Jung proposes model error models which allow a combination of simple nominal models and provide a better visualization of the possible deficiencies of the nominal model.

In [4], S. Prajna develop new methods for validation of continuous-time non-linear systems. They use Barrier certificates whose existence proves inconsistency of a model, from experimental data. But, construction of these Barrier certificates is, in general, not easy.

In a previous paper, we presented an identification of black-box model [1].
By assuming that the input/output functional is causal analytical according to M.Fliess [5], we presented the identification process of the unknown system's generating series up to an order k, and the resulting bilinear model (B_k).

To validate a continuous-time model of an unknown dynamic system, we propose an exact symbolic computation of the error due to approximation by (B_k), described by the series (G_k) and global modeling of an unknown dynamical system (Σ).
This computation is done in the case of a single input with drift, for a certain differential equations class enclosing the electric equation. We use a deterministic model (versus probabilistic one) by considering

that data noises are bounded.
So, the purpose of this paper is to apply combinatorial techniques for computing outputs of bilinear systems, and their differences at order k and k-1.
This error computation is a sum of differential monomials in the input functions and behavior system. We identify each differential monomial with its colored multiplicity and analyse our computation in the light of the free differential calculus.
This error computation allows one to better measure the impact of noisy inputs on the convergence of (B_k). Indeed, one can determine the contribution of the inputs and of the system in the error computation. The outline of this paper is as follows. In section 2, we briefly recall this model construction.
Section 3 is devoted to computational aspects. In particular, our symbolic computation is based on state equations produced by the bilinear systems (B_k).
Computing the coefficients of generating series (G_k) would be easier but incomplete because they are used for describing the system's behavior and they don't give information about system's input.
Application examples are given in section 4. Finally, we present a combinatorial interpretation of coefficients of generating series.

2 Modeling of a dynamic system by a bilinear systems family

The problem consists in modelling an unknown dynamic system (Σ) for $t \in [0, T] = \bigcup_{i \in I} [t_i, t_i + d]$, when knowing some correlated sets of input/output.
We construct a behavioral model, based on the identification of its input/output functional (the generating series), in a neighborhood of every t_i, up to a given order k [1]. At once a local modeling by a bilinear system $(B_i)_k$ around every t_i is provided. Then a family $((B_i)_{i \in I})_k$, global modelling of the unknown system is produced, such that the outputs of (Σ) and $((B_i)_{i \in I})_k$ coincide up to order k.

2.1 Affine systems

We assume that the unknown dynamic sytem is an affine system:

$$(\Sigma) \qquad \begin{cases} \dot{q} &= A_0(q) + \sum_{j=1}^{m} A_j(q) u_j(t) \\ y(t) &= h(q(t)) \end{cases}$$

- $\mathbf{u}(t) = (u_j(t))_{j=1..m}$ is the input, $y(t) \in I\!R$ is the output.
- $q(t) \in \mathcal{V}$ is the state, with $\mathcal{V}$ real manifold,
- $\{A_j(q)\}_{j=1..m}$ are analytic vector fields, $h : \mathcal{V} \to I\!R$ is an analytic function (observation)

The coefficient of the generating series G is:

$$\langle G \mid z_{i_k} z_{i_{k-1}} \cdots z_{i_1} \rangle = [A_{i_1} \circ A_{i_2} \circ \cdots \circ A_{i_k} \circ h(q)]_0 \tag{1}$$

2.2 Rational series and bilinear systems

A bilinear system is a dynamic system satisfying one of the two following propositions:

1. **its generating series is rational.**
2. **its state equation only contains some finite dimension matrices.**

More precisely, a bilinear system can be defined by its state equations:

$$(B) \begin{cases} \dot{\mathrm{x}}(\mathrm{t}) &= (M_0 + \sum_{i=1}^{m} u_i(t) M_i) x(t) \\ \overline{y}(t) &= \lambda . x(t) \end{cases} \tag{2}$$

such that

- $x(t) \in I\!R-$vector space Q of finite dimension.
- $\{M_i\}_{0 \leq i \leq m} : Q \to Q$ and $\lambda : Q \to I\!R$ are $I\!R-$linear.

2.3 Identification of the generating series in a neighborhood of t_i

The problem consists in identifying the generating series G_k up to an order k, when we know the Taylor expansions of the system inputs (around t_i) and the Taylor expansions of the corresponding outputs (around t_i).
The computing of the generating series coefficients consists in solving some judicious linear equations systems [6].

3 Particular dynamical system

In the case of a single input with drift, we consider a certain class $(\mathcal{G}P)$ enclosing the electric equation

$$y^{(1)}(t) + f(y(t)) = u(t) \tag{3}$$

where $u(t)$ is the input function
In this case, equation (3) can be written

$$(\Sigma) \qquad \left\{ \begin{array}{rcl} \dot{q} & = & A_0(q) + A_1(q)u(t) \\ y(t) & = & q(t) \end{array} \right.$$

- u(t) is the input function
- q(t) is the current state
- $A_0 = a^{(0)} \frac{d}{dq}$ where $\quad a^{(0)} = f(q)|_{q(0)}$
- $A_1 = \frac{d}{dq}$

By using state equations, we obtain a symbolic computation of the output's difference at order k and k-1. We have shown in [1] that the limit of systems (B_k) outputs is in fact the exact system output. So, by majorization of these output's differences, and when k tends towards infinity, we can get an overestimation of the error due to approximation by the (B_k).
The approximative bilinear system (B_k), at order k, is given by

$$\left\{ \begin{array}{rcl} \dot{x}_k(t) & = & (M_0 + M_1 u(t))x_k(t) \\ \overline{y}_k(t) & = & \lambda x_k(t) \end{array} \right.$$

where $\lambda = (q(0) \quad 1 \quad 0 \cdots 0)$

$$x_k(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$M_0 = (C_{z_0 z_1^k})$ (resp $M_1 = (C_{z_1^{k+1}})$) expressed in basis $(C_{z_1^k})$.

$$M_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ a^{(0)} & a^{(1)} & a^{(2)} & \cdots & a^{(k)} \\ 0 & a^{(0)} & 2a^{(1)} & \cdots & 0 \\ 0 & 0 & a^{(0)} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

So, at order k, we may obtain the ith derivative of the state vector x as a function of the previous ones.

4 Symbolic computation of difference of two consecutive outputs

We analyze these equations in the light of the free differential calculus. Considering the derivative $a^{(i)}$ and $u^{(i)}$ specialized in time t=0 as differential letters, it is clear that our computation is a sum of differential monomials in a and u.

4.1 Colored partitions and multiplicities

A number partition or multiplicity is a sequence $\mu = (\mu_1, \mu_2, \mu_3, \cdots)$ (often written as $1^{\mu_1}2^{\mu_2}3^{\mu_3}\cdots$) of nonnegative integers. On a single letter a, the differential monomials become :

$$a^\mu = (a^{(i_1)})^{e_1}(a^{(i_2)})^{e_2}\cdots(a^{(i_q)})^{e_q}, \quad 1 \leq i_1 < i_2 < \ldots i_q$$

Such a monomial is indexed by the following partition [8] :

$$\mu = (i_1^{\mu_{i_1}} i_2^{\mu_{i_2}} \cdots i_q^{\mu_{i_q}})$$

Let $C = \{a, u\}$ be a set of two colors. We call colored partition on C an element of the free monoid generated by the cartesian product $N \times N$ i.e. any finite sequence of couples of nonnegative integers

$$\mu = ((\mu_1^a, \mu_1^u), (\mu_2^a, \mu_2^u), \cdots)$$

So, a colored partition μ will denote the differential monomial

$$a^\mu = (a^{(i_1)})^{e_1}\cdots(a^{(i_p)})^{e_p}(u^{(j_1)})^{f_1}\cdots(u^{(j_q)})^{f_q}$$

$$1 \leq i_1 < i_2 < \ldots i_p, \quad 1 \leq j_1 < i_2 < \ldots j_q$$

where e_l (resp f_l) $= \mu_{i_l}^a$ (resp $\mu_{j_l}^u$). The weight and the size of μ are defined as follows :

$$wgt(\mu) = \sum_c \sum_k k.\mu_k^c$$

$$size(\mu) = \sum_c \sum_k \mu_k^c$$

The empty partition is noted ϵ.
If L is the set of colored partitions, we define a partial order $\ll$ on L :

$$\nu = \{(\nu_i^a, \nu_i^u)\} \ll \mu = \{(\mu_i^a, \mu_i^u)\}$$

if

$$\nu_i^a \leq \mu_i^a \quad and \quad \nu_i^u \leq \mu_i^u \quad \forall i$$

L, with this partial ordering forms a Young lattice. [9]
We consider now B_i a subset of L defined by :

$$\{\mu / wgt(\mu) = i\}$$

and we note $I(\mu_{max})$ the order ideal generated by μ_{max}, if

$$\mu_{max} = max(\mu/\mu \in B_i)$$

4.2 Combinatorial analysis of our computation

Let us now interpret combinatorially our computation by identifying each differential monomial with its colored multiplicity. The recursive relation is captured by the operation :

$$\mu_{max} \odot c = \sum_{\substack{\nu \in I(\mu_{max}) \\ wgt(\nu)=j\leq i}} c^{(i-j+1)}.\nu$$

By factorizing according to the colored partitions, we get :

$$x_{(k-n)k}^{(k-n+i)} = \sum_{c} \sum_{\substack{\nu \in I(\mu_{max}) \\ wgt(\nu)=j \leq i}} c^{(i-j+1)}.\nu.g^{1}_{(c^{(i-j+1)}\nu)}$$

where :

$$g^{l}_{a^{(m)}\nu} = (a^{(0)} + u^{(0)})^{m+1} \sum_{p=m}^{n_l+m} \binom{l}{m} g_v^p$$

and

$$g^{l}_{u^{(m)}\nu} = (a^{(0)} + u^{(0)})^{m-1} \sum_{p=1}^{n_l} \binom{l+i+1}{m} g_v^p$$

with $n_1 = k-n-1, \quad n_l = l \quad \forall l > 1$

$$g_\epsilon = 1$$

4.3 Computation of $x_{(k-n)k}^{k-n+i}$

We consider now permutations of a colored partition μ on an alphabet $X = \bigcup_{c \in C} X_c$. A permutation [9] of μ is a word in which each letter belongs to X and for each $x_i \in X$, the total number of appearances of x_i in the word is μ_i^c, for some $c \in C$
Let us note $\pi = \xi_1\xi_2 \cdots \xi_{size(\mu)}$ a permutation of μ and σ_μ the set of permutations of μ.
Since, our alphabet $X_a = \{a^{(p)} | p = 1, min(k-1, i+1)\}$,and $X_u = \{u^{(p)} | p = 1, i+1)\} \quad \xi_j = c^{(i_j)}$, for some c,$i_j$.
$x_{(k-n)k}^{(k-n+i)}$ is a linear combination of monomial $y_1^{\lambda_1} \cdots y_n^{\lambda_n} (y_i \in X_a \bigcup X_u)$ and all distinct monomials obtained from it by a permutation of variables.
We get finally , if $s = (\sum_j j| \quad \mu_j^u \neq 0)$ and r =size(μ)

$$x_{(k-n)k}^{(k-n+i)} = \sum_{wgt(\mu)=i+1} \mu.(a^{(0)} + u^{(0)})^{k-n+i-r-s} \quad g_\mu^n \tag{4}$$

$$g_\mu^n = \sum_{\pi \in \sigma_\mu} A_1 \prod_{j=2}^{r} A_j + b \tag{5}$$

where:

$$A_j = \begin{cases} \sum_{m_j=i_j}^{m_{j-1}+i_j} \binom{m_j}{i_j} & if \quad \xi_j = a^{(i_j)} \\ \sum_{m_j=1}^{m_{j-1}} \binom{m_j+i-j+2}{i_j} & if \quad \xi_j = u^{(i_j)} \end{cases}$$

$$A_1 = \begin{cases} \sum_{m_1=m}^{k-n-2+m} \binom{m_1}{i_1} & if \quad \xi_1 = a^{(i_1)} \\ \sum_{m_j=1}^{k-n-2} \binom{m_1+i+1}{i_1} & if \quad \xi_1 = u^{(i_1)} \end{cases}$$

and b = 1 if $\xi_1 = u^{(i+1)}$, 0 otherwise.

4.4 Computation of difference of state vectors

The first derivative coincide up to order k-2, but at order k-1, we have
$x_{2k}^{(k-1)} - x_{2(k-1)}^{(k-1)} = 0$ and $x_{jk}^{(k-1)} - x_{j(k-1)}^{(k-1)} \neq 0$.
Let M (resp P) the set of partitions on the single letter a (resp u)
W_i a subset of M defined by

$$\{\nu | 1 \leq size(\nu) \leq i+2\}$$

V_i a subset of P defined by

$$\{\lambda | size(\lambda) = \lfloor \frac{i}{2} \rfloor, wgt(\lambda) \leq i-2 \quad or \quad \lambda = u^{(i-2)} \quad or \quad \lambda = u^{(i-1)}\}$$

and S_l a subset of L defined by

$$\{\mu | wgt(\mu) = l\}$$

We define now an operation $\nabla : M \times P \times L \mapsto L$

$$\nabla(\nu, \lambda, \mu) = ((\nu_i + \mu_i^a, \lambda_i + \mu_i^c))_i$$

and a subset P_t of $L \quad \forall 0 \leq t \leq i$

$$P_t = \{\tau = \nabla(\nu, \lambda, \mu) | \quad \mu \in S_t, \lambda \in V_i, \nu \in W_i, wgt(\tau) = k + i - 1\}$$

We obtain, by a straightforward computation :

$$x_{2k}^{(k+i)} - x_{2(k-1)}^{(k+i)} = \sum_{\substack{\nabla(\nu,\lambda,\mu) \in P_t \\ 0 \leq t \leq i}} \nabla(\nu, \lambda, \mu) \quad h_\nu.f_\lambda.g_\mu^1.(a^{(0)} + u^{(0)})^{k+i-2-r_1-s}$$

where

$$f_\lambda = \sum_{\pi \in \sigma_\lambda} \prod_{l=1}^{size(\lambda)} \binom{k + i - 2l}{k + i - 2l - i_l}$$

$$h_\nu = \begin{cases} \sum_{\pi \in \sigma_\nu^1} \prod_{j=1}^{r-2} \binom{i_j + i_{j+1} - 1}{i_{j+1}} \binom{k-2}{i_r} & if \quad size(\nu) \neq 1 \\ 1 & if \quad size(\nu) = 1 \end{cases}$$

with r = size(ν), $r_1 = r + size(\mu)$, s = $(\sum_j j | \mu_j^u \neq 0)$ $\pi = \xi_1 \xi_2 \cdots \xi_r$, $\xi_j = c^{(i_j)}$, g_μ^1 defined previously.

$$\sigma_\nu^1 = \{\pi \in \sigma_\nu | \pi \neq \nu_1.\mu, size(\nu_1) < size(\nu) \quad and \quad \pi \neq (a^{(1)})^{r-1}.\xi_r\}$$

- (i) We remove permutations π which could be equal to a partition $\tau = \nu_1.\mu$, for some τ
- (ii) We remove also permutations which begin by the first derivative of letter a.

4.5 Computation of the error $(\overline{\mathbf{y}}_\mathbf{k}(\mathbf{t}) - \overline{\mathbf{y}}_{\mathbf{k-1}}(\mathbf{t}))$

The (k-1) first derivative of the two outputs coincide at point t=0. So, using Taylor expansion, we can write

$$\overline{y}_k(t) - \overline{y}_{k-1}(t) = \sum_{i \geq k} (\overline{y}_k^{(i)}(0) - \overline{y}_{k-1}^{(i)}(0)).\frac{t^i}{i!}$$

Taking into account that $\overline{y}_k^{(i)}(0) = x_{2k}^{(i)}(0)$, we obtain a right computation of the output's difference at order k and k-1.

5 Numerical examples

Our examples are based on the computation of the two consecutive outputs at order 2 and 3 (we note $E3$ this error), in the case of equations like electric equation. Taylor expansion is at order 5. So we get :

$$\overline{y}_3(t) - \overline{y}_2(t)$$

$$= \frac{t^3.a_2.(a_0 + u(0))^2}{3!} + \frac{t^4(q_{33}^{(4)}(0) - q_{23}^{(4)}(0))}{4!} + \frac{t^5(q_{33}^{(5)}(0) - q_{23}^{(5)}(0))}{5!} + \epsilon(t)$$

where

$$\epsilon(t) = O(t^6)$$

and

$$(q_{33}^{(4)}(0) - q_{23}^{(4)}(0) =$$
$$2a_1a_2(a_0 + u(0))^2 + 3a_2(a_0 + u(0))u^{(1)}(0)$$

and

$$(q_{33}^{(5)}(0) - q_{23}^{(5)}(0) = 3a_1a_2(a_0 + u(0))^2 +$$
$$7a_1a_2(a_0 + u(0))u^{(1)}(0) +$$
$$a_2^2(a_0 + u(0))^3 +$$
$$4a_2(a_0 + u(0))u^{(2)}(0) +$$
$$3a_2(u^{(1)})^2(0)$$

We want to point up three facts, through these examples :

- The error is too important beyond some interval and our model is not "acceptable"
- The error depends on system behavior and its stability
- The error is different from smooth inputs to rough inputs

5.1 First examples

We try in this example to measure the impact of the system. We show in [11] that the system is bounded input bounded output (BIBO) for $a^{(1)} \leq 0$ and not BIBO for $a^{(1)} > 0$. So, we took $k_1 = k_2 = 1$ in the previous example ($a^{(1)} \leq 0$ for these values of parameters).
In the array below, the parameters are $k_1 = k_2 = -1$ ($a^{(1)} > 0$). The input function is the same (sin(t)).
We see that the error is more important when the system is not stable (not BIBO).

t	E3
0.1	0.0016
0.5	0.41
1.0	6.65
1.5	38.54
2.	140.8

Table 1: First Table .

5.2 Second examples

We measure here the impact of input functions. We take $i(t) = 100 * t^6 + 10$. The parameters of the system are $k_1 = k_2 = 1$(Case of stability of the system).

For an input function $i(t) = 100 * t^6 + 1000$, the error is about 10^9.

t	E3
0.1	- 0.0201
0.5	- 1.7
1.0	-3.4
1.5	19.47
2.	211.2

Table 2: Second Table .

6 Conclusion

We have proposed, in this paper, a validation of our model for a particular class of dynamic systems. This validation is not statistical. It consists in valuing the convergence of a bilinear models family (B_k) on the unknown system (Σ) by an effective symbolic computation.
It allows to determine intervals where our model is "'acceptable"', as shown with numerical examples. It displays the respective contributions of the input and of the system itself.
More than a symbolic validation, these computing tools are parameterized by the input and the system's behavior. They can particularly provide a valuation process for rough and oscillating inputs as well as for smooth inputs.

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