A LINEARLY IMPLICIT FINITE ELEMENT METHOD FOR A KLEIN–GORDON–SCHRÖDINGER-TYPE SYSTEM

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ABSTRACT

Let $T > 0$, $\Omega := (x_A, x_B) \subset \mathbb{R}$ be a bounded interval and $D := [0, T] \times \overline{\Omega}$. Then, we consider the following initial- and boundary- value problem for a Klein–Gordon–Schrödinger (DKGS) system of partial differential equations: find functions $\psi : D \to \mathbb{C}$ and $\phi : D \to \mathbb{R}$ solving the following nonlinear system of equations

$$\psi_t = i\,\mu\,\psi_{xx} - \alpha\,\psi - i\,\phi\,\psi \quad \text{on} \quad (0, T] \times \Omega, \tag{0.1}$$

$$\phi_{tt} = \phi_{xx} - \phi - \lambda\,\phi_t - \mathrm{Re}(\psi_x) \quad \text{on} \quad (0, T] \times \Omega, \tag{0.2}$$

and satisfying the conditions

$$\psi(t, x) = 0 \quad \text{and} \quad \phi(t, x) = 0 \quad \forall\,(t, x) \in [0, T] \times \partial\Omega, \tag{0.3}$$

$$\psi(0, x) = \psi_0(x) \quad \forall\,x \in \overline{\Omega}, \tag{0.4}$$

$$\phi(0, x) = \phi_0(x) \quad \text{and} \quad \phi_t(0, x) = \phi_1(x) \quad \forall\,x \in \overline{\Omega}, \tag{0.5}$$

where: $\mu > 0$, $\alpha > 0$, $\lambda > 0$ are known real numbers, and ϕ_0, $\phi_1 : \overline{\Omega} \to \mathbb{R}$ and $\psi_0 : \overline{\Omega} \to \mathbb{C}$ are given functions. The DKGS system models the nonlinear interaction between high frequency electron waves and low frequency ion plasma waves in a homogeneous magnetic field. Such systems arise in the UHH plasma heating technique. The coupling of the two fluids is achieved through the (non-homogeneous) polarization drift and the induced current takes part in the collisional process of energy exchange. As a consequence, the nonlinearity differs from the one encountered in the Zakharov system [5]. For a detailed derivation of the model, along with the underlying assumptions, the reader is referred to [1] and [4]. Also, we refer to [1] for results on the existence and uniqueness of a solution of the problem. In the present work, we focus on the derivation of an efficient numerical method to approximate the solution of the initial and boundary value problem (0.1)-(0.5).

Let $r \in \mathbb{N}$ and $S_h^r \subset H_0^1(\Omega)$ be a finite element space consisting of functions which are piecewise polynomials of degree at most r over a partition of Ω in intervals with maximum mesh-length h. It is well-known that the following approximation property holds: there exists a constant $C_{A,r} > 0$ such that

$$\inf_{\chi \in S_h^r} \left\{ \|v - \chi\|_{L^2(\Omega)} + h\,\|v - \chi\|_{H^1(\Omega)} \right\} \leq C_{A,r}\,h^\ell\,\|v\|_{H^\ell(\Omega)} \quad \forall\,v \in H^\ell(\Omega) \cap H_0^1(\Omega), \quad \ell = 1, \dots, r + 1. \tag{0.6}$$

Then, we define the discrete Laplacian operator $\Delta_h : H^1(\Omega) \to S_h^r$ by $(\Delta_h \varphi, \chi)_{L^2(\Omega)} = (\varphi', \chi')_{L^2(\Omega)}$ $\forall\,\chi \in S_h^r$, $\forall\,\varphi \in H^1(\Omega)$, and the L^2-projection operator $P_h : L^2(D) \to S_h^r$ by $(P_h f - f, \chi)_{L^2(\Omega)} = 0$ $\forall\,\chi \in S_h^r$, $\forall\,f \in L^2(D)$. Also, we introduce elliptic projection operators R_h^Ψ, $R_h^\Phi : H_0^1(\Omega) \to S_h^r$ by $((R_h^\Psi v - v)', \chi')_{L^2(\Omega)} = 0\ \forall\,\chi \in S_h^r, \forall\,v \in H_0^1(\Omega)$, and $((R_h^\Phi v - v)', \chi')_{L^2(\Omega)} + (R_h^\Phi v - v, \chi)_{L^2(\Omega)} = 0\ \forall\,\chi \in S_h^r$, $\forall\,v \in H_0^1(\Omega)$. Now, let $R_h = R_h^\Psi$ or R_h^Φ. The approximation property (0.6) (see [2]), yields

$$\|R_h v - \chi\|_{L^2(\Omega)} + h\,\|R_h v - \chi\|_{H^1(\Omega)} \leq C_{B,r}\,h^\ell\,\|v\|_{H^\ell(\Omega)} \quad \forall\,v \in H^\ell(\Omega) \cap H_0^1(\Omega), \quad \ell = 1, \dots, r + 1, \tag{0.7}$$

Also, from [3] it holds that

$$\|R_h v - v\|_{L^\infty(\Omega)} \leq C_{\Gamma,r}\,h^\ell\,\|v\|_{W^{\ell,\infty}(\Omega)} \quad \forall\,v \in W^{\ell,\infty}(\Omega) \cap H_0^1(\Omega), \quad \ell = 1, \dots, r + 1. \tag{0.8}$$

Let $N \in \mathbb{N}$, $k := \frac{T}{N}$ and $t^m := m\,k$ for $m = 0, \dots, N$. The proposed method constructs, for $m = 0, \dots, N$, an approximation $(\Psi_h^m, \Phi_h^m) \in S_h^r \times S_h^r$ of $(\psi(t^m, \cdot), \phi(t^m, \cdot))$ following the steps below:

`Step 1:` Set

$$\Psi_h^0 := R_h^{\Psi}\psi_0 \quad \text{and} \quad \Phi_h^0 := R_h^{\Phi}\phi_0. \tag{0.9}$$

`Step 2:` Set

$$\Phi_h^1 := R_h^{\Phi}\big\{\phi_0 + k\,\phi_1 + \tfrac{k^2}{2}\,\big[\,\phi_0'' - \phi_0 - \lambda\,\phi_1 - \mathrm{Re}(\psi_0')\,\big]\big\}. \tag{0.10}$$

`Step 3:` Find $\Psi_h^1 \in S_h$ such that

$$\tfrac{\Psi_h^1 - \Psi_h^0}{k} = i\,\mu\,\Delta_h\Big(\tfrac{\Psi_h^1 + \Psi_h^0}{2}\Big) - \alpha\,\tfrac{\Psi_h^1 + \Psi_h^0}{2} - i\,P_h\Big[\tfrac{\Phi_h^1 + \Phi_h^0}{2}\,\tfrac{\Psi_h^1 + \Psi_h^0}{2}\Big]. \tag{0.11}$$

`Step 4:` For $n = 1, \dots, N-1$, specify $(\Psi_h^{n+1}, \Phi_h^{n+1}) \in S_h \times S_h$ via the requirements

$$\tfrac{\Psi_h^{n+1} - \Psi_h^{n-1}}{2k} = i\,\mu\,\Delta_h\Big(\tfrac{\Psi_h^{n+1} + \Psi_h^{n-1}}{2}\Big) - \alpha\,\tfrac{\Psi_h^{n+1} + \Psi_h^{n-1}}{2} - i\,P_h\Big[\Phi_h^n\,\tfrac{\Psi_h^{n+1} + \Psi_h^{n-1}}{2}\Big] \tag{0.12}$$

and

$$\tfrac{\Phi_h^{n+1} - 2\Phi_h^n + \Phi_h^{n-1}}{k^2} = \Delta_h\Big(\tfrac{\Phi_h^{n+1} + \Phi_h^{n-1}}{2}\Big) - \lambda\,\tfrac{\Phi_h^{n+1} - \Phi_h^{n-1}}{2k} - \tfrac{\Phi_h^{n+1} + \Phi_h^{n-1}}{2} - P_h\Big[\,\mathrm{Re}\big(\,(\Psi_h^n)'\,\big)\Big]. \tag{0.13}$$

Analyzing the numerical method above, first we ensure that it is well-defined without mesh conditions, and then, we derive an a priori bound for the numerical approximations, in a discrete energy norm, described below

$$\max_{0 \le m \le N}\{\,\|\Phi_h^m\|_1 + \|\Psi_h^m\|_1\,\} + \max_{0 \le n \le N-1}\|\partial_k\Phi_h^n\|_0 \le C\Big[\|\partial_k\Phi_h^0\|_0 + \|\Phi_h^1\|_1 + \|\Phi_h^0\|_1 + \|\Psi_h^0\|_1 + \|\Psi_h^0\|_0^3\Big]. \tag{0.14}$$

Finally, we combine (0.14), (0.6), (0.7) and (0.8), with a proper stability argument, to arrive at an optimal order error estimate in the L^2 and H^1 norms, i.e.,

$$\max_{0 \le m \le N}\Big\{\|\Psi_h^m - \psi^m\|_{H^1(\Omega)} + \|\Phi_h^m - \phi^m\|_{H^1(\Omega)}\Big\} \le C\,(k^2 + h^r)$$

and

$$\max_{0 \le m \le N}\Big\{\|\Psi_h^m - \psi^m\|_{L^2(\Omega)} + \|\Phi_h^m - \phi^m\|_{L^2(\Omega)}\Big\} \le C\,(k^2 + h^{r+1}).$$

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