Stable Computation of the Optimal Path for a Boat on a Water Stream and Its Applications

Kokichi Sugihara and Tetsushi Nishida
Department of Mathematical Informatics
University of Tokyo, Tokyo 113-8656, Japan

We present a stable method for computing the optimal path for a boat in a continuous flow field, and apply it to the simulations of diffusion phenomena such as accidental diffusion of petrol from an oil tanker in ocean.

Consider a boat that can move at an isotropic speed on still water, where "isotropic" means that the boat can move at the same speed in any direction if there is no flow of water; the speed of the boat may depend on the location of the boat. Suppose that we are given a continuous flow field of water and a pair of a start point and a goal. We are interested in finding the optimal path in the sense of the shortest traveling time for a boat to move from the start point to the goal.

Let $f(x, y)$ be a two-dimensional flow vector at point $(x, y) \in \mathbf{R}^2$. Suppose that the boat starts at point p_0 at time 0, and let $E(t)$ be the boundary of the region which the boat can reach by time t. We denote by $p(s, t)$ a point on the curve $E(t)$, where s is a parameter such that $p(s, t)$ moves along $E(t)$ as s changes.

Then, $p(s, t)$ obeys the next equation:

$$\frac{\partial p(s,t)}{\partial t} = Fn(s,t) + f(p(s,t)), \tag{1}$$

where $n(s, t)$ denotes the normal to the curve $E(t)$ at $p(s, t)$ and F denotes the speed of the boat on still water.

A typical method to solve the equation (1) might be a marker particle method [2]. In this method we consider many particles that are placed at the start point at time 0, and move in different directions according to the equation, hoping that the particles at time t might be distributed along the curve $E(t)$. However, this scheme is not stable. Indeed the normal $n(s, t)$ in the equation (1) is estimated by the neighboring two particles, but this estimation is unreliable because the curve $E(t)$ is not necessarily differentiable. Note that there can be two or more optimal paths from the start point to some special points, and hence the curve $E(t)$ has singular points. Therefore in order to trace the particles in a stable manner, we have to estimate the normal $n(s,t)$ without consulting the neighboring particles.

Fortunately we can prove that the normal $n(s,t)$ satisfies the next equation [1]:

$$\frac{\partial n(s,t)}{\partial t} = (I - n(s,t)n(s,t)^{\mathrm{T}})J^{\mathrm{T}}(\bigtriangledown^{\mathrm{T}} f)Jn(s,t), \tag{2}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \bigtriangledown^{\mathrm{T}} f = \left(\frac{\partial f(x,y)}{\partial x} \quad \frac{\partial f(x,y)}{\partial y} \right). \tag{3}$$

Note that in this equation the time derivative of the normal is expressed by only $n(s,t)$ and $\bigtriangledown^{\mathrm{T}} f$, and hence the time change of the normal can be computed without consulting the neighboring particles.

Therefore, using the equations (1) and (3), we can trace each particle independently, that is, we can trace it in a robust manner even if the particle passes through neighbors of singular points. In other words, the curve $\{p(s,t) \mid s = \text{const.}\}$ is always non-singular while the curve $\{p(s,t) \mid t = \text{const.}\}$ may have singular points. We call our new method an "independent marker particle method".

The independent marker particle method can be applied to the simulation of oil diffusion in a given ocean flow. Actually if the oil tanker comes across an accident at p_0 at time 0 and oil starts leaking out, then $E(t)$ represents the frontier curve of the oil diffusion area.

The equations (1) and (3) can also be used for the estimation of the source of diffusion from the frontier curve $E(t)$ at some time t. Indeed, if we reverse the orientation of the normal $n(s,t)$ and also replace the time parameter t by $-t$, then the same equations (1) and (3) hold. Therefore, the same system of equations can represents the behavior of the particles in the part, and hence we can trace the particles back to the past. By this scheme, we can estimate the location and the time at which the oil tanker came across the accident.

References

[1] T. Nishida, K. Sugihara and M. Kimura: Stable marker-particle method for a Voronoi diagram in a flow field. *Journal of Computational and Applied Mathematics* (to appear).

[2] W. Rider and D. Kothe: A marker particle method for interface tracking. *Proceedings of the 6th International Symposium on Computational Fluid Dynamics*, 1995, pp. 976–981.