# On the Iterative Analysis of the Generalized Dirichlet-Neumann Map for Elliptic PDEs*

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**Abstract.** Taking advantage of the structural properties of the Collocation coefficient matrix associated with the Dirichlet-Neumann map for linear elliptic PDEs, we present a complete spectral analysis for the Laplace's equation on a square domain with the same type of boundary conditions on all sides. Through this analysis we are able to recover optimal classical SOR and Krylov iterative methods.

## 1 Introduction

Recently, Fokas[1, 4] introduced a new unified approach for analyzing linear and integrable nonlinear PDEs. Central issue to this approach is a generalized Dirichlet to Neumann map, characterized through the solution of the so-called *global relation*, namely, an equation, valid for all values of a complex parameter $k$, coupling specified known and unknown values of the solution and its derivatives on the boundary. In particular, for the case of Laplace's equation, $q_{z\bar{z}} = 0$, in a convex bounded polygon $D$ with vertices $z_1, z_2, \ldots, z_n$ (modulo $n$) indexed counter-clockwise, the associated *Global Relation* takes the form (see also [2, 3])

$$\sum_{j=1}^{n} \int_{S_j} e^{-ikz} q_z dz = 0, \quad k \in \mathbb{C}\ , \tag{1}$$

where $S_j$ denotes the side from $z_j$ to $z_{j+1}$ (not including the end points). If, for $z \in S_j,\ \ 1 \le j \le n$, we now let $g^{(j)}$ denote the derivative of the solution in the direction making an angle $\beta_j, 0 \le \beta_j \le \pi$ with the side $S_j$, namely : $\cos(\beta_j)\, q_s^{(j)} + \sin(\beta_j)\, q_n^{(j)} = g^{(j)}$, and $f^{(j)}$ denote the derivative of the solution in the direction normal to the above direction, namely : $-\sin(\beta_j)\, q_s^{(j)} + \cos(\beta_j)\, q_n^{(j)} = f^{(j)}$, where $q_s^{(j)}$ and $q_n^{(j)}$ denote the tangential and (outward) normal components of $q_z$ along the side $S_j$, then the *Generalized Dirichlet-Neumann map*, that is the relation between the sets $\left\{f^{(j)}(s)\right\}$ and $\left\{g^{(j)}(s)\right\}_{j=1}^{n}$, is characterized by the single equation

$$\sum_{j=1}^{n} |h_j|\, e^{i(\beta_j - k m_j)} \int_{-\pi}^{\pi} e^{-ikh_j s} \left(f^{(j)} - ig^{(j)}\right) ds = 0, \qquad k \in \mathbb{C} \tag{2}$$

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