Numerical solution of a non-local elliptic problem modelling a thermistor with a finite element and a finite volume method

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Abstract

Let $D := (-1,1)$, and the following non-local elliptic boundary value problem:

$$w''(x) + \lambda \frac{f(w(x))}{\left(\int_D f(w)dx\right)^2} = 0 \quad \forall x \in D, \tag{1}$$

$$w'(1) + a\,w(1) = 0, \qquad w'(-1) - a\,w(-1) = 0, \tag{2}$$

where $w = w(x;\lambda)$ and λ is a dimensionless parameter. The problem (1)-(2) models the steady state temperature profile of the thermistor device. We will focus in the case of the Negative Temperature Coefficient thermistor (NTC-thermistor), where the electrical resistivity decreases with temperature, e.g. $f(s) = e^{-s}$ or $f(s) = (1+s)^{-p}$. It is has been proved that if $f(s) > 0$, $f'(s) < 0$, $f''(s) > 0$ for $s > 0$, and $\int_0^\infty f(s)\,ds < \infty$, then the problem (1)-(2), has at least one classical (regular) solution for the critical value of the parameter λ, say λ^*, has no solution for $\lambda > \lambda^*$, and for $\lambda < \lambda^*$ attains at least two regular solutions ($\overline{w}$, $\underline{w}$) where $\underline{w}(x) < \overline{w}(x)$ for $x \in D$, $\underline{w}$ is stable and $\overline{w}$ is unstable for λ close to λ^*. In addition we may scale f so that $\int_0^\infty f(s)\,ds = 1$, and then $\lambda^* < 8$. For the equation (1) but with Dirichlet b.c.'s, $w(\pm 1) = 0$, we have that $\lambda^* = 8$ (if $\int_0^\infty f(s)ds = 1$). For $\lambda < \lambda^*$ we have a unique stable solution w, while for $\lambda \geq \lambda^*$ we have no solution. For $f(s) = e^{-s}$ the analytical solution for problem (1)-(2) is known. It holds that $w(x) = \frac{2\gamma}{\alpha}\tan(\gamma) + 2\ln(\frac{\cos(\gamma x)}{\cos(\gamma)})$ where γ solves the equation $\lambda = 8\sin^2(\gamma)\exp(-\frac{2\gamma}{\alpha}\tan(\gamma))$, for λ, a known. Also for $\alpha = 1$, λ^* can be computed and is found to be $\lambda^* \simeq 1.1239$. When the Dirichlet b.c.'s, $w(\pm 1) = 0$, imposed to equation (1) with $f(s) = e^{-s}$, then $w(x) = 2\ln\left[\frac{\cos(\gamma x)}{\cos(\gamma)}\right]$ for $\gamma = \sin^{-1}(\frac{\lambda}{8})^{\frac{1}{2}}$. These analytical solutions can be used for the comparison with the numerical results presented in this work. The accurate knowledge for the steady solution is needed, in order to obtain estimates for the evolutionary problem and for the practical point of view of applications.

In the paper at hand, in order to approximate the solution of (1) with Robin or Dirichlet b.c.'s, we construct a finite element and a finite volume method based on piecewise continuous piecewise quadratic functions. In particular, the proposed finite volume method extends a new finite volume method derived recently in a work by M. Plexousakis and G.E. Zouraris for general linear two-point boundary value problems. Both methods leads to a nonlinear system of algebraic equations that we solve by applying an iterative method. In the case of the Robin boundary conditions (2), when we start the iterative method below this solution, e.g., from zero, it is expected that we approximate the solution belonging to the stable branch of the response diagram (minimal solution $\underline{w}$) which is the situation of interest regarding the application of the model. Apart this it is useful to compare the finite element method and the finite volume method for a nonlinear elliptic problem, since the general theory for finite volume methods is not as extensive as for the finite element methods.

We consider a partition of D with $J+1$ nodes $\{x_j\}_{j=0}^J$ where $J \geq 3$, $x_0 = -1$, $x_J = 1$ and $x_j < x_{j+1}$ for $j = 0, \dots, J-1$. Then, set $I_j := (x_{j-1}, x_j)$ for $j = 1, \dots, J$, $x_{j+z} := x_j + z\,(x_{j+1} - x_j)$ for $j = 0, \dots, J-1$ and $z \in [0,1]$, and $\xi_j : I_j \to [0,1]$ by $\xi_j(x) := \frac{x - x_{j-1}}{x_j - x_{j-1}}$ for $x \in I_j$ and $j = 1, \dots, J$. Let $I = (y_L, y_R)$. Then, we denote by $\mathcal{X}_I$ the characteristic function of the interval I, and we write $[\![v]\!]_{\partial I} = v(y_R^-) - v(y_L^+)$, where $v(x^\pm) := \lim_{\epsilon \to 0^+} v(x \pm \epsilon)$.

The methods we propose construct an approximation of the solution of problem (1)-(2) from the space $S^2_{\mathcal{R}}$ consisting of functions which are continuous on $[-1,1]$ and reduce to polynomials of degree less than or equal to 2 on each $I \in \{I_j\}_{j=1}^J$. When we consider the equation (1) with Dirichlet boundary conditions the methods construct an approximations of the solution from the space $S^2_{\mathcal{D}} := \{\phi \in S^2_{\mathcal{R}} : \phi(\pm 1) = 0\}$. We note that $\dim(S^2_{\mathcal{R}}) = 2J+1$ and $\dim(S^2_{\mathcal{D}}) = 2J-1$. Let $\rho \in (0,1)$ be a real parameter, and $\{\Delta_j\}_{j=1}^{2J+1}$ be control volumes given by $\Delta_{2\ell} := (x_{\ell-1}, x_\ell)$ for $j = 1, \dots, J$, $\Delta_{2\ell+1} := (x_{\ell-1+\rho}, x_{\ell+\rho})$ for $j = 1, \dots, J-1$, $\Delta_1 := (x_0, x_{0+\rho})$ and $\Delta_{2J+1} := (x_{J-1+\rho}, x_J)$. The proposed finite volume method (FVM) is formulated as

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follows: find $w_h \in S^2_{\mathcal{R}}$ such that

$$\begin{aligned}
- \quad & \Big[w_h'(x_{0+\rho}) - a\, w_h(x_0) \Big] = V(w_h;\Delta_1),\\
- \quad & \Big[w_h'(x_1) - a\, w_h(x_0) \Big] = V(w_h;\Delta_2),\\
- \quad & [\![w_h']\!]_{\partial\Delta_j} = V(w_h;\Delta_j), \qquad j = 3,\ldots,2J-1,\\
- \quad & \Big[-a\, w_h(x_J) - w_h'(x_{J-1}) \Big] = V(w_h;\Delta_{2J}),\\
- \quad & \Big[-a\, w_h(x_J) - w_h'(x_{J-1+\rho}) \Big] = V(w_h;\Delta_{2J+1})
\end{aligned} \tag{3}$$

where $V(w_h;\Delta) := \lambda \frac{\int_\Delta f(w_h(x))\,dx}{\left(\int_D f(w_h(x))\,dx\right)^2}$. Using the auxiliary functions $\widehat{\phi}_0(x) := \frac{2-3\rho}{1-\rho}(1-x^2) + \frac{2\rho-1}{1-\rho}(1-x)$, $\widehat{\phi}_{\frac{1}{2}}(x) := 6x(1-x)$, $\widehat{\phi}_1(x) := \frac{3\rho-1}{\rho}x^2 + \frac{1-2\rho}{\rho}x$, we construct a basis $\mathcal{B}^{FV}_{\mathcal{R}} = \{\varphi_j\}_{j=1}^{J+1} \cup \{\varphi_{j-\frac{1}{2}}\}_{j=1}^{J}$ of $S^2_{\mathcal{R}}$ by $\varphi_1(x) = \widehat{\phi}_0(\xi_1(x))\,\mathcal{X}_{I_1}(x)$, $\varphi_{j-\frac{1}{2}}(x) = \widehat{\phi}_{\frac{1}{2}}(\xi_j(x))\,\mathcal{X}_{I_j}$ for $j = 1,\ldots,J$, $\varphi_j(x) = \widehat{\phi}_1(\xi_{j-1}(x))\,\mathcal{X}_{I_{j-1}}(x) + \widehat{\phi}_0(\xi_j(x))\,\mathcal{X}_{I_j}(x)$ for $j = 2,\ldots,J$, and $\varphi_{J+1}(x) = \widehat{\phi}_1(\xi_J(x))\,\mathcal{X}_{I_J}(x)$. Hence, $w_h = \sum_{i=1}^{J+1} \beta_i^{FV}\,\varphi_i + \sum_{i=1}^{J} \beta_{J+1+i}^{FV}\,\varphi_{i-\frac{1}{2}}$, where $\beta = \{\beta_i^{FV}\}_{i=1}^{2J+1}$ is the coefficients vector to be determined. Then, (3) is equivalent to a nonlinear system of algebraic equations of the form $A^{FV}\beta^{FV} = F^{FV}(\beta^{FV})$, where $A^{FV} \in \mathbf{R}^{(2J+1)\times(2J+1)}$ is a matrix and $F^{FV} : \mathbf{R}^{2J+1} \to \mathbf{R}^{2J+1}$ is a nonlinear map defined by $(F^{FV}(y))_i := V\Big(\sum_{j=1}^{J+1} y_j\,\varphi_j + \sum_{j=1}^{J} y_{J+1+j}\,\varphi_{j-\frac{1}{2}};\Delta_i \Big)$ for $y \in \mathbf{R}^{2J+1}$ and $i = 1,\ldots,2J+1$. We solve the obtained nonlinear system by an iterative process based on Broyden's method with initial approximation $\beta^{FV}_{(1)} = 0 \in \mathbf{R}^{2J+1}$. When, we consider the equation (1) with Dirichlet boundary conditions, the finite volume method is formulated as follows: find $w_h \in S^2_{\mathcal{D}}$ such that:

$$-[\![w_h']\!]_{\partial\Delta_j} = V(w_h;\Delta_j), \quad j = 2,\ldots,2J.$$

To formulate the analogous nonlinear system of algebraic equations, we choose the basis $\mathcal{B}^{FV}_{\mathcal{D}} = \{\varphi_j\}_{j=2}^{J} \cup \{\varphi_{j-\frac{1}{2}}\}_{j=1}^{J}$ of $S^2_{\mathcal{D}}$. In the numerical experiments, we choose $\rho = \frac{1}{2} - \frac{\sqrt{3}}{6}$ because, this is one of the values which, in the linear case, ensure an optimal order of convergence in the L^2, H^1 and L^∞ norms.

The finite element method (FEM) for problem (1)-(2) is formulated as follows: find $w_h \in S^2_{\mathcal{R}}$ such that

$$a\Big[w_h(1)\,\phi(1) + w_h(-1)\,\phi(-1) \Big] + (w_h', \phi')_{0,D} = W(w_h,\phi) \quad \forall\,\phi \in S^2_{\mathcal{R}}, \tag{4}$$

where $W(w_h,\phi) := \lambda \frac{(f(w_h),\phi)_{0,D}}{\left(\int_D f(w_h)\,dx\right)^2}$. In the numerical experiments we use a basis $\mathcal{B}^{FE}_{\mathcal{R}} = \{\phi_j\}_{j=1}^{2J+1}$ of $S^2_{\mathcal{R}}$ determined by $\phi_{J+1+j}(x) = \widehat{\phi}_{\frac{1}{2}}(\xi_j(x))\mathcal{X}_{I_j}(x)$ for $j = 1,\ldots,J$, $\phi_1(x) = \widehat{\phi}_0(\xi_1(x))\,\mathcal{X}_{I_1}(x)$, $\phi_j(x) = \widehat{\phi}_1(\xi_{j-1}(x))\,\mathcal{X}_{I_{j-1}}(x) + \widehat{\phi}_0(\xi_j(x))\mathcal{X}_{I_j}$ for $j = 2,\ldots,J$, and $\phi_{J+1} = \widehat{\phi}_1(\xi_J(x))\,\mathcal{X}_{I_J}$, where $\widehat{\phi}_0(x) := 1-x$, $\widehat{\phi}_{\frac{1}{2}}(x) := 4\,x\,(1-x)$ and $\widehat{\phi}_1(x) := x$. Thus $w_h = \sum_{i=1}^{2J+1} \beta_i^{FE}\,\phi_i$, where $\beta^{FE} = \{\beta_i^{FE}\}_{i=1}^{2J+1}$ is the coefficients vector to be specified. It is easily seen that (4) is equivalent to a nonlinear system of algebraic equations of the form $A^{FE}\beta^{FE} = F^{FE}(\beta^{FE})$, where $A^{FE} \in \mathbf{R}^{(2J+1)\times(2J+1)}$ is a matrix and $F^{FE} : \mathbf{R}^{2J+1} \to \mathbf{R}^{2J+1}$ is a nonlinear map defined by $(F^{FE}(y))_i := W(\sum_{j=1}^{2J+1} y_j\,\phi_j, \phi_i)$ for $y \in \mathbf{R}^{2J+1}$ and $i = 1,\ldots,2J+1$. As in the FVM the resulting nonlinear system is solved by an iterative process based on Broyden's method. When we consider the equation (1) with Dirichlet boundary conditions, the FEM is formulated as follows: find $w_h \in S^2_{\mathcal{D}}$ such that:

$$(w_h', \phi')_{0,D} = W(w_h,\phi) \quad \forall\,\phi \in S^2_{\mathcal{D}}.$$

The corresponding nonlinear system of algebraic equations, is obtained choosing the basis $\mathcal{B}^{FE}_{\mathcal{D}} = \{\phi_j\}_{j=2}^{J} \cup \{\phi_{j-\frac{1}{2}}\}_{j=1}^{J}$ of $S^2_{\mathcal{D}}$.

The problem was solved numerically on a uniform grid consisting of $J+1 = M = 20$, 40, 80, or 160 grid points and using tolerance TOL= 10^{-10} in the Newton-type method. Also, we choose $a = 1$, $\lambda = 1$ and $f(s) = e^{-s}$, i.e., λ is chosen so that $\lambda < \lambda^*$. The L^2 and H^1 norms of the error $w - w_h$ were computed using Simpson's rule, and the L^∞ norm of the error was estimated by a finite sampling at the abscissae of the aforementioned quadrature rule. The results indicate that the computational order of convergence agrees with the order of convergence in the linear case, which is equal to 3 in the L^2 and L^∞ norms, and 2 in the H^1 norm.

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