Tail Gini's Risk Measures and Related Linear Programming Models for Portfolio Optimization

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Following the seminal work by Markowitz (1952), the portfolio optimization problem is modeled as a mean-risk bicriteria optimization problem where the expected return is maximized and some (scalar) risk measure is minimized. In the original Markowitz model the risk is measured by the standard deviation or variance. Several other risk measures have been later considered thus creating the entire family of mean-risk (Markowitz-type) models. While the original Markowitz model forms a quadratic programming problem, following Sharpe (1971), many attempts have been made to linearize the portfolio optimization procedure (c.f., Speranza (1993) and references therein). The Linear Programming (LP) solvability is very important for applications to real-life financial decisions where the constructed portfolios have to meet numerous side constraints (including the minimum transaction lots, transaction costs and mutual funds characteristics). The introduction of these features leads to mixed integer LP problems. In order to guarantee that the portfolio takes advantage of diversification, no risk measure can be a linear function of the portfolio weights. Nevertheless, a risk measure can be LP computable in the case of discrete random variables, i.e., in the case of returns defined by their realizations under specified scenarios.

The simplest LP computable risk measures are dispersion measures similar to the variance. Yitzhaki (1982) introduced the mean-risk model using Gini's mean (absolute) difference as the risk measure. The Gini's mean difference (GMD) turn out to be a special aggregation technique of the multiple criteria LP model (Ogryczak, 2000) based on the pointwise comparison of the absolute Lorenz curves. The latter leads to the quantile shortfall risk measures which are more commonly used and accepted. Recently, the second order quantile risk measures have been introduced in different ways by many authors. The measure, now commonly called the Conditional Value at Risk (CVaR) (after Rockafellar and Uryasev, 2000) or Tail VaR, represents the mean shortfall at a specified confidence level. It leads to LP solvable portfolio optimization models in the case of discrete random variables represented by their realizations under specified scenarios. The CVaR measures maximization is consistent with the second degree stochastic dominance as shown by Ogryczak and Ruszczyński (2002). Several empirical analyses confirm its applicability to various financial optimization problems. Thus, the CVaR models seem to overstep the measure of Value-at-Risk (VaR) defined as the maximum loss at a specified confidence level which is commonly used in banking.

Although any CVaR measure is risk relevant, it represents only the mean within a part (tail) of the distribution of returns. Therefore, such a single criterion is in some manner crude for modeling various risk aversion preferences. The Gini's mean difference model combines all CVaR measures averaging all shortfalls. In order to enrich the modeling capabilities, one needs to treat differently some more or less extreme events. This require some techniques to enhance the downside risk aversion (Krzemienowski and Ogryczak, 2005). Alternatively, the Tail GMD measures may be applied (Ogryczak and Ruszczyński, 2002), which averages the shortfall within specified quantiles. The latter may approximated with appropriate combinations of multiple CVaR measures.

In this paper we study LP solvable portfolio optimization models based on the use of multiple CVaR measures thus allowing for more detailed risk aversion modeling. All the studied models are shown to be SSD consistent. They may be considered some approximations to the Gini's mean difference with the advantage of being computationally much simpler than the GMD model itself. Our analysis has been focused on the Weighted CVaR measures defined as simple combinations of a very few CVaR measures. We have introduced two specific types of weight-settings which relate the WCVaR measure to the Gini's mean difference (the Wide WCVaR) and its tail version (the Tail WCVaR). This allows us to use a few tolerance levels as only parameters specifying the entire WCVaR measures while the corresponding weights are automatically predefined by the requirements of the corresponding Gini's measures. Theoretical properties and computational efficiency of the models is studied as well as their practical achievements are tested.

Our experimental analysis of the models performance on the real-life data from the Milan Stock Exchange has confirmed their attractiveness. The WCVaR models have usually performed better than the GMD, the Minimax or the extremal CVaR models. These promising results show a need for further comprehensive experimental studies analyzing practical performances of the WCVaR models within specific areas of financial applications. It is important to notice that although the quantile risk measures (VaR and CVaR) were introduced in banking as extreme risk measures for small tolerance levels (like 5%), for the portfolio optimization good results have been provided by rather larger tolerance levels. Additional experimental analysis over the period with strongly negative market trend has confirmed good achievements of the WCVaR models. They have, on average, performed better than the GMD and the extremal CVaR models. In terms of cumulative returns all the models have outperformed the MIB30 index.

While the Tail WCVaR models have always generated well diversified portfolios, the Wide WCVaR models require some diversification enforcement to avoid too small portfolios. Our experiments have also confirmed effectiveness of our CVaR based technique for a direct diversification enforcement. Although, the diversification enforcement turns out to be necessary and effective rather during unstable market trends (typically characterized by quick changes of market directions) than during strong downward periods.

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