Calculating the volume of n-dimensional spheres

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Abstract

In this paper, my aim is to show a procedure for calculating of volume and the boundary "surface area" of n-dimensional spheres. By appalling a simple method, we are trying to generalize the definition of "sphere" from three-dimensions to n-dimension. The next step here is to introduce a simple way to proof of generated equations by understanding some equations in order to calculate it.

1- Introduction.

As we know, the set of points in space at a given distance from a fixed point is a sphere. Here fixed point is the center and the given distance, is the radius "r" of the sphere. This definition does not depend on the number of the dimensions of the space.

Therefore we define an infinite dimensional sphere with n-dimensional space which can be assumed as n-dimensional sphere.

Obviously, the cross-section of a zero-dimensional sphere is zero, because a dimensionless space is defined only by a singular point.

$$x^2 = 0 \qquad (1)$$

But in any one-dimensional sphere, it is includes of two points which can show as a straight line. (fig 1).

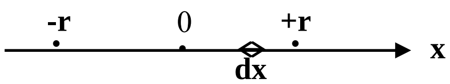

Fig.1 : One-dimensional sphere

From here, the cross-section of one-dimensional sphere can be as follow,

$$x^2 = r^2 \qquad (2)$$

Moreover, we can determine the volume of a one-dimensional sphere from the following equation.

$$\int_{-r}^{+r} dx = 2r \qquad (3)$$

Here as we can see, the variation of x for a one-dimensional sphere, is between -r and +r .

Furtherer in a two-dimensional sphere, which in fact is known as a circle, (fig 2)

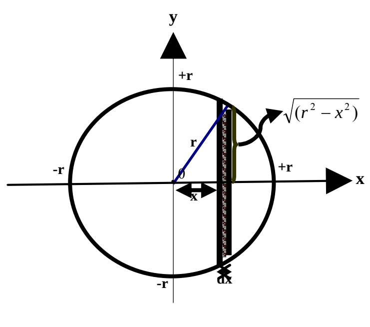

Fig.2 : Two-dimensional sphere

its equation is,

$$x^2 + y^2 = r^2 \qquad (4)$$

By choosing the area element (fig 2) and equation (3), we can find out the volume of two-dimensional sphere as follow.

$$\int_{-r}^{+r} 2(r^2 - x^2)^{\frac{1}{2}}\, dx = \pi r^2 \tag{5}$$

Consequently for a three-dimensional sphere, (fig 3)

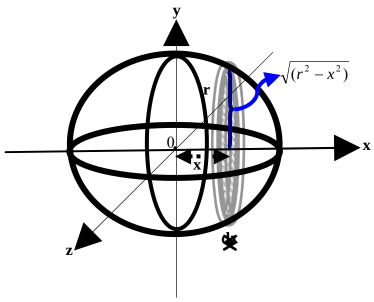

Fig.3 : Three-dimensional sphere

which has the equation of $x^2 + y^2 + z^2 = r^2$, then we can obtain,

$$\int_{-r}^{+r} \pi[(r^2 - x^2)^{\frac{1}{2}}]^2\, dx = \frac{4}{3}\pi r^3 \tag{6}$$

And in the higher dimensions

Four-dimensional sphere $$\int_{-r}^{+r} \frac{4}{3}\pi[(r^2 - x^2)^{\frac{1}{2}}]^3\, dx = \frac{1}{2}\pi^2 r^4 \tag{7}$$

Five-dimensional sphere $$\int_{-r}^{+r} \frac{1}{2}\pi^2[(r^2 - x^2)^{\frac{1}{2}}]^4\, dx = \frac{8}{15}\pi^2 r^5 \tag{8}$$

Six-dimensional sphere $$\int_{-r}^{+r} \frac{8}{15}\pi^2[(r^2 - x^2)^{\frac{1}{2}}]^5\, dx = \frac{1}{6}\pi^3 r^6 \tag{9}$$

Seven-dimensional sphere $$\int_{-r}^{+r} \frac{1}{6}\pi^3[(r^2 - x^2)^{\frac{1}{2}}]^6\, dx = \frac{16}{105}\pi^3 r^7 \tag{10}$$

and so on. Therefore we are able to find out, the volume of any n-dimensional sphere. By using this method it is clear that the volume of n-dimensional spheres, can be derive from two separated equations as follow,

$$\frac{\pi^{(\frac{n}{2})}}{(\frac{n}{2})!} r^n \qquad \text{(n - even)} \tag{11-a}$$

$$\frac{2^{(\frac{n+1}{2})}\pi^{(\frac{n-1}{2})}}{n!!} r^n \qquad \text{(n - odd)} \tag{11-b}$$

where here n is an integer number. In proof of the above equations (11-a) and (11-b), we can consider as following proof.

2- Proof

If **n** is even: n=2k, k=1,2,3,4,… We have to prove that:

$$\int_{-r}^{+r} \frac{2^{[\frac{(n-1)+1}{2}]}\pi^{[\frac{(n-1)-1}{2}]}}{(n-1)!!}[(r^2 - x^2)^{\frac{1}{2}}]^n\, dx = \frac{\pi^{(\frac{n}{2})}}{(\frac{n}{2})!} r^n \tag{12}$$

But because **n** is even, we can assume **n** equal to 2k, then the left-hand side will be:

$$\frac{2^k \pi^{(k-1)}}{(2k-1)!!}\int_{-r}^{+r}[r^2 - x^2]^{(\frac{2k-1}{2})}dx \tag{13}$$

By the substitution of $x = r\sin\theta$, we will have:

$$\frac{2^k \pi^{(k-1)} r^{2k}}{(2k-1)!!}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos^{2k}\theta d\theta \tag{14}$$

By applying the equation below:

$$\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos^{n}\theta d\theta = (\frac{n-1}{n})\int_{\frac{-\pi}{2}}^{\frac{+\pi}{2}}\cos^{n-2}\theta d\theta \tag{15}$$

We will have:

$$= \frac{2^k \pi^{(k-1)} r^{2k}}{(2k-1)!!}\times\frac{2k-1}{2k}\times\frac{2k-3}{2k-2}\times\frac{2k-5}{2k-4}\times\cdots\times\frac{1}{2}\times\pi \tag{16}$$

$$= \frac{2^k \pi^{(k-1)} r^{2k}}{(2k-1)!!}\times\frac{\pi(2k-1)!!}{2^k k!} = \frac{\pi^k}{k!}r^{2k} = \frac{\pi^{(\frac{n}{2})}}{(\frac{n}{2})!}r^n \tag{17}$$

And if **n** is odd: n=2k-1, k=1,2,3,4,… We have to prove that:

$$\int_{-r}^{+r}\frac{\pi^{(\frac{n-1}{2})}}{(\frac{n-1}{2})!}[(r^2 - x^2)^{\frac{1}{2}}]^{(n-1)}dx = \frac{2^{(\frac{n+1}{2})}\pi^{(\frac{n-1}{2})}}{n!!}r^n \tag{18}$$

But because **n** is even, we can assume **n** equal to 2k-1, then the left-hand side will be:

$$\frac{\pi^{(k-1)}}{(k-1)!}\int_{-r}^{+r}[r^2 - x^2]^{(k-1)}dx = \frac{\pi^{(k-1)} r^{(2k-1)}}{(k-1)!}\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\cos^{2k-1}\theta d\theta \tag{19}$$

$$= \frac{\pi^{(k-1)} r^{(2k-1)}}{(k-1)!}\times\frac{2k-2}{2k-1}\times\frac{2k-4}{2k-3}\times\frac{2k-6}{2k-5}\times\cdots\times 2 \tag{20}$$

$$= \frac{\pi^{(k-1)} r^{(2k-1)}}{(k-1)!}\times\frac{2^k (k-1)!}{(2k-1)!!} = \frac{2^k \pi^{(k-1)} r^{(2k-1)}}{(2k-1)!!} = \frac{2^{(\frac{n+1}{2})}\pi^{(\frac{n-1}{2})}}{n!!}r^n \tag{21}$$

Here the equations (17) and (21) are the same as equations of (11-a) and (11-b) respectively.

3- Conclusion

From here we can find out that the boundary (or surface area) of an n-dimensional sphere, produced by differentiating of its volume respected to radius r.

Otherwise, the boundary "surface area" of any n-dimensional sphere multiplied by (n/r) will be the volume of its n-dimensional sphere.

Summaries of results

Dimension (n)	volume	surface area
1	2r	2
2	πr^2	$2\pi r$
3	$(4/3)\pi r^3$	$4\pi r^2$
4	$(1/2)\pi^2 r^4$	$2\pi^2 r^3$
5	$(8/15)\pi^2 r^5$	$(8/3)\pi^2 r^4$
6	$(1/6)\pi^3 r^6$	$\pi^3 r^5$
7	$(16/105)\pi^3 r^7$	$(16/15)\pi^3 r^6$
8	$(1/24)\pi^4 r^8$	$(1/3)\pi^4 r^7$
9	$(32/945)\pi^4 r^9$	$(32/15)\pi^4 r^8$
10	$(1/120)\pi^5 r^{10}$	$(1/12)\pi^5 r^9$

Table.1.

References:

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