Computation of the Generalized Mittag-Leffler Function and its Inverse in the Complex Plane

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The generalized Mittag-Leffler function $\mathrm{E}_{\alpha,\beta}(z)$ has been computed for arbitrary complex argument $z \in \mathbb{C}$ and parameters $\alpha \in \mathbb{R}^+$ and $\beta \in \mathbb{R}$ [1]. This function plays a fundamental role in the theory of fractional differential equations and numerous applications in physics. The Mittag-Leffler function interpolates smoothly between exponential and algebraic functional behaviour. A numerical algorithm for its evaluation has been developed. The algorithm is based on integral representations and exponential asymptotics. Results of extensive numerical calculations are presented. We find that all complex zeros emerge from the point $z = 1$ for small α. They diverge towards $-\infty + (2k-1)\pi\mathrm{i}$ for $\alpha \to 1^-$ and towards $-\infty + 2k\pi\mathrm{i}$ for $\alpha \to 1^+$ ($k \in \mathbb{Z}$). All complex zeros collapse pairwise onto the negative real axis for $\alpha \to 2$. We introduce and study also the inverse generalized Mittag-Leffler function, and determine its principal branch numerically.

[1] R. Hilfer and H.J. Seybold, Integral Transforms and Special Functions, vol. 17, (2006), p. 637