A discrete Adomian decomposition method for the discrete nonlinear Schrödinger equation

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In this work we want to describe a discrete version of the well–known Adomian decomposition method (ADM) applied to the discrete nonlinear Schrödinger equation (NLS). The ADM was introduced by Adomian [3], [11] in the early 1980s to solve nonlinear ordinary and partial differential equations. The discrete nonlinear Schrödinger equation is omnipresent [8] in applied sciences, e.g. describing the propagation of electromagnetic waves in glass fibers, one–dimensional arrays of coupled optical waveguides [6] and light–induced photonic crystal lattices [5]. Moreover, it is used to describe Bose–Einstein condensates in optical lattices [10] and it is an established model for optical pulse propagation in various doped fibers [7]. Specifically, we will consider the two most common discrete versions of the standard cubic NLS equation that arise from different spatial discretizations. These discrete nonlinear Schrödinger equations (DNLS) are also called lattice NLS equation [9, Chapter 5.2.2]. For an extension of the ADM to the fully discrete NLS, i.e. after a discretization in time we refer the reader to [4].

The nonlinear cubic Schrödinger equation (NLS) is a typical dispersive nonlinear partial differential equation that plays a key role in a variety of areas in mathematical physics. It describes the spatio–temporal evolution of the complex field $u = u(x,t) \in \mathcal{C}$ and has the general form

$$\begin{aligned} i\partial_t u + \partial_x^2 u + q|u|^2 u &= 0, \quad x \in \Re,\ t > 0, & (1)\\ u(x,0) &= f(x), & (2) \end{aligned}$$

where the parameter $q \in \Re$ corresponds to a focusing ($q > 0$) or defocusing ($q < 0$) effect of the nonlinearity. Applying the standard spatial discretization to standard cubic NLS equation and replacing $F(u) = |u|^2 u$ with a diagonal discretization $F_D(u_j) = |u_j|^2 u_j$, we obtain the usual DNLS equation

$$\begin{aligned} i\partial_t u_j + D_h^2 u_j + q\,|u_j|^2\,u_j &= 0, \quad j \in \mathcal{Z},\ t > 0, & (3)\\ u_j(0) &= f_j,\ j \in \mathcal{Z} & (4) \end{aligned}$$

with $u_j = u_j(t)$, $h = \Delta x$ and $D_h^2 u_j = (u_{j+1} - 2u_j + u_{j-1})/h^2$ to denote the standard second order difference quotient. The parameter $\varepsilon := h^{-2}$ is called discrete dispersion and the parameter q is called anharmonicity, since equation (3) with $\varepsilon = 0$ describes a set of uncoupled anharmonic oscillators.

The DNLS equation (3) has a discrete conserved number (mass, total excitation norm, power in nonlinear optics)

$$N_D = \frac{2}{q} \sum_{j \in \mathcal{Z}} |u_j|^2 \tag{5}$$

and the discrete Hamiltonian

$$H_D = -\sum_{j \in \mathcal{Z}} \Big[u_j^*(u_{j+1} + u_{j-1}) - 2|u_j|^2 + \frac{q}{2}|u_j|^4 \Big], \tag{6}$$

where * denotes the complex conjugate.

After a discretization in space by replacing the cubic nonlinearity $F(u) = |u|^2 u$ in standard cubic NLS equation with an off–diagonal discretization $F_{AL}(u_j) = |u_j|^2(u_{j+1} + u_{j-1})/2$ and keeping the time variable continuous we obtain the Ablowitz–Ladik (AL) equation [1], [2]

$$\begin{aligned} i\,\partial_t\, u_j + D_h^2\, u_j + q|u_j|^2\, \frac{u_{j+1} + u_{j-1}}{2} &= 0,\ t > 0, & (7)\\ u_j(0) &= f_j,\ j \in \mathcal{Z}. & (8) \end{aligned}$$

Note that one term in equation (7) can be removed through the transformation $u_j(t) = v_j(t) \exp(-i\,2\,t)$; $t > 0$ and equation (7) reduces to the normalized form

$$i\partial_t\, v_j + \frac{v_{j+1} + v_{j-1}}{h^2}$$

$$+q|v_j|^2\, \frac{v_{j+1} + v_{j-1}}{2} = 0,\ j \in \mathcal{Z},\ t > 0. \tag{9}$$

The AL equation has a conserved number

$$N_{AL} = \frac{2}{q} \sum_{j \in \mathcal{Z}} \log\Big(1 + \frac{q}{2}|u_j|^2\Big) \tag{10}$$

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and the Hamiltonian

$$\begin{aligned} H_{AL} &= -\sum_{j\in\mathcal{Z}} \Big[u_j^*(u_{j+1}+u_{j-1}) \\ &\qquad -\frac{4}{q}\log\Big(1+\frac{q}{2}|u_j|^2\Big)\Big]. \end{aligned} \tag{11}$$

Now the analogue discrete steps to the continuous ADM of the standard cubic NLS equation are simply the formal solution to the DNLS equation (3) or the AL equation (7)

$$\begin{aligned} u_j(t) &= f_j + iL_t^{-1}D_h^2 u_j \\ &\quad + iqL_t^{-1}F_{D,AL}(u_j),\ j\in\mathcal{Z},\ t>0, \end{aligned} \tag{12}$$

and the assumption that there exists a solution of the series form $u_j(t)=\sum_{l=0}^{\infty}u_{j,l}(t)$. The nonlinear term $F_{D,AL}(u_j)$ in (12) is decomposed into an infinite series of discrete Adomian polynomials

$$F_{D,AL}(u_j)=\sum_{l=0}^{\infty}A_l(u_j). \tag{13}$$

Substituting (13) decompositions into (12) gives

$$\sum_{l=0}^{\infty}u_{j,l}(t)=f_j+i\sum_{l=0}^{\infty}L_t^{-1}D_h^2u_j(t)+iq\sum_{l=0}^{\infty}L_t^{-1}A_l. \tag{14}$$

Again, $u_{j,0}(t)$ is identified with the initial data f_j and the following recurrence is proposed to determine the solution components $u_{j,l}(t)$

$$u_{j,0}(t) = f_j, \tag{15}$$

$$\begin{aligned} u_{j,l+1}(t) &= iL_t^{-1}D_h^2u_{j,l}(t) \\ &\quad + iqL_t^{-1}A_l\,;\ \ l=0,1,2,\ldots \end{aligned} \tag{16}$$

The polynomials A_l can be computed in a tedious calculation. The calculations to obtain the Adomian polynomials were performed using `Mathematica`.

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