A fourth order numerical scheme for the sine-Gordon equation

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Abstract— **A numerical method arising from the use of rational approximants to the matrix-exponential term in a three-time level recurrence relation is proposed for the numerical solution of the one-dimensional sine-Gordon equation already known from the bibliography. The proposed method uses a second order scheme for the predictor and a fourth order one for the corrector. To avoid extended matrix evaluations in the implementation of the corrector an auxiliary vector has been introduced successfully. Both the predictor and the corrector schemes are analysed for stability. The method has been tested on the single soliton and conclusions with corresponding results known in the bibliography are derived.**

Keywords—**Soliton; Sine-Gordon; Finite-difference method; Predictor-corrector.**

AMS — 35Q51, 35Q53, 65M06, 78M20, 65Y10.

I. Introduction

Since Russell's [11] first observation for some special waves with characteristic properties on a canal in 1834 a lot of scientists among whom Korteweg & de Vries [9] have investigated them extensively. The name *soliton*, which takes its original meaning from the ancient Greek word όν, was first used in Zabusky and Kruskal [12] in order to emphasize that a soliton is a localized entity which keeps its identity after interaction. Equations which also lead to solitary waves are the sine-Gordon and the cubic Schrödinger equation (see for extended details in [1]).

The one-dimensional SG equation, which has the general form

$$u_{tt} = u_{xx} - \sin u\,;\;\; L_0 < x < L_1\,,\; t > t_0, \qquad (1.1)$$

with $u = u\,(x,t)$ a sufficiently differentiable function, has been investigated theoretically by mathematicians of the last century like Enneper, Eisenhart, Darboux, Bianci (see for example [7]) etc. The SG equation is a particular case $V'(u) = \sin u$ of the Klein-Gordon equation $u_{tt} = u_{xx} - V'(u)$ in which $V'(u) = d\,V(u)/du$. It first appeared in differential geometry, while other important physical phenomena in problems including the dislocation theory in crystals, the propagation in ferromagnetic materials of waves carrying rotations of the direction of magnetization, the laser pulses in two state media etc.

The SG equation has also been analysed numerically by means of finite-difference (FDM), finite-element, pseudo spectral, Adomian decomposition method etc. As far as FDM they can be found, among other publications, in Guo et *al* [8], Ramos [10], Bratsos and Twizell [2], Bratsos [3]-[4] etc.

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Initial conditions will be assumed to have the form

$$u\,(x,t_0) = f\,(x) \text{ and } \left.u_t\right|_{t=t_0} = g\,(x)\;;\;\; L_0 \le x \le L_1, \qquad (1.2)$$

while the boundary conditions

$$\left.u_x\right|_{x=L_0} = \left.u_x\right|_{x=L_1} = 0\,;\quad t > t_0. \qquad (1.3)$$

It is known in the bibliography that the SG equation has soliton solutions which can be expressed as follows

$$u\,(x,t) = 4\tan^{-1}\left\{\pm\exp\left[\pm\gamma\,(x - c\,t) + a\right]\right\}, \qquad (1.4)$$

where c is the velocity with $c^2 < 1$, a is a constant and $\gamma = \left(1 - c^2\right)^{-1/2}$. The four possible sign combinations in Eq. (1.4) lead to *kinks* if the signs are similar and to *antikinks* when the signs are opposite. Finally, the energy E (see for example [8] etc.) given by

$$E(t) = \frac{1}{2}\int_{-\infty}^{+\infty}\left[u_x^2 + u_t^2 + 2\,(1 - \cos u)\right]\,dx \qquad (1.5)$$

is conserved.

II. The numerical method

A. Development of the method

In order to obtain a numerical solution the region $R = [L_0 < x < L_1] \times [t > t_0]$ with its boundary $\partial\,R$ is covered with a rectangular mesh, G, of points with coordinates $(x,t) = (x_m, t_n) = (L_0 + mh, t_0 + nl)$ with $m = 0, 1, ..., N+1$ and $n = 0, 1, \ldots$. The numerical approximation at the typical mesh point of G will be denoted with U_m^n. Let the solution vector at time level $t = t_n = n\ell$ be

$$\mathbf{U}^n = \left[U_0^n, U_1^n, ..., U_{N+1}^n\right]^T. \qquad (2.1)$$

Replacing the space derivative in Eq. (1.1) with second-order central-difference approximant and applying Eq. (1.1) to all $N+2$ mesh points of the grid G at time level $t = n\ell$; $n = 0, 1, ...$, leads to a second-order initial-value problem of the form

$$\begin{aligned} D^2\,\mathbf{U}\,(t) &= A\mathbf{U}\,(t) - \mathbf{G}\,(\mathbf{U}\,(t)) \\ \mathbf{U}\,(t_0) &= \mathbf{f}\,,\, D\mathbf{U}\,(t_0) = \mathbf{g}\,; t > t_0 \end{aligned} \qquad (2.2)$$

TABLE I

Method	Expression of Eq. (2.3)
I	$\mathbf{U}(t+\ell)+\mathbf{U}(t-\ell)=\left(2\tilde{I}+\ell^2D^2\right)\mathbf{U}(t)$
II	$\left(I-\frac{1}{12}\ell^2D^2+\frac{1}{144}\ell^4D^4\right)[\mathbf{U}(t+\ell)+\mathbf{U}(t-\ell)]$ $=\left(2I+\frac{5}{6}\ell^2D^2+\frac{1}{72}\ell^4D^4\right)\mathbf{U}(t)$

in which $D = \mathrm{diag}\{d/dt\}$, $D^2 = \mathrm{diag}\left\{d^2/dt^2\right\}$ are diagonal matrices of order $N+2$, $\mathbf{G}(\mathbf{U}(t)) = \left[\sin U_0^n, \sin U_1^n, ..., \sin U_{N+1}^n\right]^T$ is a vector of order $N+2$ and A is a tridiagonal matrix of order $N+2$ with appropriate entries.

The numerical methods will be developed by using the three-time level recurrence relation

$$\mathbf{U}(t+\ell) = [\exp(\ell D)+\exp(-\ell D)]\,\mathbf{U}(t) - \mathbf{U}(t-\ell),\ t=\ell, 2\ell, ... \tag{2.3}$$

in which the matrix-exponential term is replaced with rational or Padé approximants. The expression of Eq. (2.3) for the methods which are going to be examined in this paper is given in Table I. Except for the explicit Method I the implicit Method II finally leads to the solution of a nonlinear system of the form

$$\mathbf{F}(\mathbf{U}(t+\ell)) = \mathbf{0}. \tag{2.4}$$

B. The Predictor-Corrector scheme

To avoid solving the nonlinear system (2.4) the following *Predictor-Corrector* (P-C) scheme is proposed.

B.1 Predictor

The value $\hat{\mathbf{U}}(t+\ell)$ is evaluated from Method I, whose relevant expression in Table I is written as

$$\hat{\mathbf{U}}(t+\ell) = \left(2I+\ell^2D^2\right)\mathbf{U}(t) - \mathbf{U}(t-\ell), \tag{2.5}$$

where $D^2\,\mathbf{U}(t)$ is given by Eq. (2.2).

B.2 Corrector

The following explicit to the unknown vector $\mathbf{U}(t+\ell)$ scheme arising from Method II is proposed

$$\begin{aligned}\mathbf{U}(t+\ell) &= 2\mathbf{U}(t) + \frac{1}{12}\ell^2D^2\left[\hat{\mathbf{U}}(t+\ell)+10\mathbf{U}(t)\right.\\ &\quad \left.+\mathbf{U}(t-\ell)\right] - \frac{1}{12}\ell^2D^2\left\{\frac{1}{12}\ell^2D^2\left[\hat{\mathbf{U}}(t+\ell)\right.\right.\\ &\quad \left.\left.-2\mathbf{U}(t)+\mathbf{U}(t-\ell)\right]\right\} - \mathbf{U}(t-\ell)\\ &= 2\mathbf{U}(t) + \frac{1}{12}\ell^2D^2\left[\hat{\mathbf{U}}(t+\ell)+10\mathbf{U}(t)\right.\\ &\quad \left.+\mathbf{U}(t-\ell)-\mathbf{V}(t)\right] - \mathbf{U}(t-\ell),\end{aligned} \tag{2.6}$$

where

$$\mathbf{V} = \mathbf{V}(t) = \left[\,V_0(t),\, V_1(t),\, ...,\, V_{N+1}(t)\,\right]^T \tag{2.7}$$

is an auxiliary vector of order $N+2$, which is used for the evaluation of the expression $D^4\,\mathbf{U}(t+\ell)$ - see Table I.

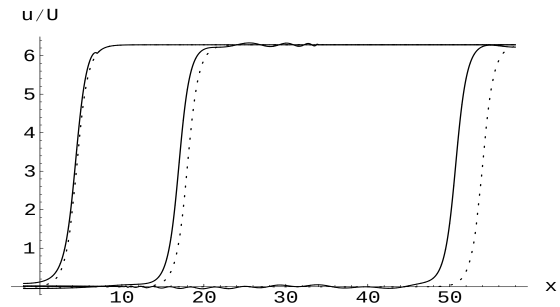

Fig. 1. Single-soliton. The dot curves show the theoretical solution u, while the full ones the numerical solution U for kinks $(+,+)$ at $t = 9,\ 36,\ 108$ when $h = 0.01$, $\ell = 0.001$ and $t \in [0, 108]$.

III. Numerical results

SG equation was solved numerically for kinks $(+,+)$ with $t \in [0, 108]$, boundary lines $L_0 = -2$, $L_1 = 58$ (see also [2]), initial conditions given by Eq. (1.2) $U(x,0) = u(x,0)$ and $U(x,\ell) = u(x,\ell)$ where u denotes the theoretical solution and the parameter values $a = 0$ and $c = 0.5$ for the theoretical solution.

From the experiments it was deduced that the energy E has an approximately constant value and that an increasing disposition between the theoretical solution u and the following it numerical U occurs (see Fig. 1).

The author is currently applying to the proposed scheme an analogous modification to that used in Bratsos [5]-[6].

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