Validation of a particular class of dynamical systems

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1 Introduction

The aim of this paper is to study the validation of a model for non-linear systems. It evaluates the ability of a model to simulate the behavior of an unknown system (Σ). This is a central problem in identification ([2]).
In almost cases, validation consists, by a statistical approach, in a test that falsifies or not falsifies the model, using a given discrete sampled data set.
Our approach, based on combinatorial techniques, is different and provides :

- an exact symbolic computation of the error E, due to the approximation of (Σ) by a family of bilinear systems described by generating series. This produces an estimation of E, that is essential in the measure of model's quality [2]. Particularly, we can determine intervals where our model is "acceptable", independently of the data set choosen for validation, as in discret case.
- a computing tool parameterized by the input and the system's behavior. In fact, our computation is a sum of differential monomials in the input functions and behavior system. So, we can determine and separate input and system contributions to this error and we obtain, in this way, a valuation process for rough and oscillating inputs as well as smooth inputs.

2 Modeling of a dynamic system by a bilinear systems family

The problem consists in modelling an unknown dynamic system (Σ) for $t \in [0,T] = \bigcup_{i \in I}[t_i, t_i + d]$, when knowing some correlated sets of input/output.
We construct a behavioral model, based on the identification of its input/output functional (the generating series), in a neighborhood of every t_i, up to a given order k [1]. At once a local modeling by a bilinear system $(B_i)_k$ around every t_i is provided. Then a family $((B_i)_{i \in I})_k$, global modelling of the unknown system is produced, such that the outputs of (Σ) and $((B_i)_{i \in I})_k$ coincide up to order k.

3 Particular class of dynamical systems

In the case of a single input with drift, we consider a certain class $(\mathcal{G}P)$ enclosing the electric equation

$$y^{(1)}(t) + f(y(t)) = u(t) \tag{1}$$

where $u(t)$ is the input function
In this case,the system (Σ) can be written

$$(\Sigma) \qquad \begin{cases} \dot{q} &= A_0(q) + A_1(q)u(t) \\ y(t) &= q(t) \end{cases}$$

- u(t) is the input function
- q(t) is the current state
- $A_0 = a^{(0)} \frac{d}{dq}$ where $\quad a^{(0)} = f(q)|_{q(0)}$
- $A_1 = \frac{d}{dq}$

The approximative bilinear system (B_k), at order k, is given by

$$\left\{\begin{array}{lcl} \dot{x}_k(t) & = & (M_0 + M_1 u(t))x_k(t) \\ \overline{y}_k(t) & = & \lambda x_k(t) \end{array}\right.$$

where $\lambda = (q(0) \quad 1 \quad 0 \cdots 0)$

$$x_k(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

By using state equations, we obtain a symbolic computation of the output's difference at order k and k-1. We have shown in [1] that the limit of systems (B_k) outputs is in fact the exact system output. So, by majorization of these output's differences, and when k tends towards infinity, we can get an overestimation of the error due to approximation by the (B_k).
We analyze these equations in the light of the free differential calculus. Considering the derivative $a^{(i)}$ and $u^{(i)}$ specialized in time t=0 as differential letters, it is clear that our computation is a sum of differential monomials in a and u.

3.1 Colored partitions and multiplicities

A number partition or multiplicity is a sequence $\mu = (\mu_1, \mu_2, \mu_3, \cdots)$ (often written as $1^{\mu_1}2^{\mu_2}3^{\mu_3}\cdots$) of nonnegative integers. On a single letter a, the differential monomials become :

$$a^\mu = (a^{(i_1)})^{e_1}(a^{(i_2)})^{e_2}\cdots(a^{(i_q)})^{e_q}, \quad 1 \le i_1 < i_2 < \ldots i_q$$

Such a monomial is indexed by the following partition [3] :

$$\mu = (i_1^{\mu_{i_1}} i_2^{\mu_{i_2}} \cdots i_q^{\mu_{i_q}})$$

Let $C = \{a, u\}$ be a set of two colors. We call colored partition on C an element of the free monoid generated by the cartesian product $N \times N$ i.e. any finite sequence of couples of nonnegative integers

$$\mu = ((\mu_1^a, \mu_1^u), (\mu_2^a, \mu_2^u), \cdots)$$

So, a colored partition μ will denote the differential monomial

$$a^\mu = (a^{(i_1)})^{e_1}\cdots(a^{(i_p)})^{e_p}(u^{(j_1)})^{f_1}\cdots(u^{(j_q)})^{f_q}$$

$$1 \le i_1 < i_2 < \ldots i_p, \quad 1 \le j_1 < i_2 < \ldots j_q$$

where e_l (resp f_l) $= \mu_{i_l}^a$ (resp $\mu_{j_l}^u$). The weight and the size of μ are defined as follows :

$$wgt(\mu) = \sum_c \sum_k k.\mu_k^c$$

$$size(\mu) = \sum_c \sum_k \mu_k^c$$

The empty partition is noted ϵ.
If L is the set of colored partitions, we define a partial order $\ll$ on L :

$$\nu = \{(\nu_i^a, \nu_i^u)\} \ll \mu = \{(\mu_i^a, \mu_i^u)\}$$

if

$$\nu_i^a \le \mu_i^a \quad and \quad \nu_i^u \le \mu_i^u \quad \forall i$$

L, with this partial ordering forms a Young lattice. [4]
We consider now B_i a subset of L defined by :

$$\{\mu / wgt(\mu) = i\}$$

and we note $I(\mu_{max})$ the order ideal generated by μ_{max}, if

$$\mu_{max} = max(\mu / \mu \in B_i)$$

3.2 Combinatorial analysis of our computation

Let us now interpret combinatorially our computation by identifying each differential monomial with its colored multiplicity. The recursive relation is captured by the operation :

$$\mu_{max} \odot c = \sum_{\substack{\nu \in I(\mu_{max}) \\ wgt(\nu) = j \leq i}} c^{(i-j+1)}.\nu$$

We consider now permutations of a colored partition μ on an alphabet $X = \bigcup_{c \in C} X_c$. A permutation [4] of μ is a word in which each letter belongs to X and for each $x_i \in X$, the total number of appearances of x_i in the word is μ_i^c, for some $c \in C$
Let us note $\pi = \xi_1 \xi_2 \cdots \xi_{size(\mu)}$ a permutation of μ and σ_μ the set of permutations of μ.
Since, our alphabet $X_a = \{a^{(p)} | p = 1, min(k-1, i+1)\}$,and $X_u = \{u^{(p)} | p = 1, i+1)\}$ $\quad \xi_j = c^{(i_j)}$, for some c,i_j.
$x_{(k-n)k}^{(k-n+i)}$ is a linear combination of monomial $y_1^{\lambda_1} \cdots y_n^{\lambda_n} (y_i \in X_a \bigcup X_u)$ and all distinct monomials obtained from it by a permutation of variables.
We get finally , if $s = (\sum_j j| \quad \mu_j^u \neq 0)$ and r =size(μ)

$$x_{(k-n)k}^{(k-n+i)} = \sum_{wgt(\mu)=i+1} \mu.(a^{(0)} + u^{(0)})^{k-n+i-r-s} \quad g_\mu^n \tag{2}$$

$$g_\mu^n = \sum_{\pi \in \sigma_\mu} A_1 \prod_{j=2}^{r} A_j + b \tag{3}$$

where:

$$A_j = \begin{cases} \sum_{m_j = i_j}^{m_{j-1}+i_j} \binom{m_j}{i_j} & if \quad \xi_j = a^{(i_j)} \\ \sum_{m_j=1}^{m_{j-1}} \binom{m_j + i - j + 2}{i_j} & if \quad \xi_j = u^{(i_j)} \end{cases}$$

$$A_1 = \begin{cases} \sum_{m_1 = m}^{k-n-2+m} \binom{m_1}{i_1} & if \quad \xi_1 = a^{(i_1)} \\ \sum_{m_j=1}^{k-n-2} \binom{m_1 + i + 1}{i_1} & if \quad \xi_1 = u^{(i_1)} \end{cases}$$

and b = 1 if $\xi_1 = u^{(i+1)}$, 0 otherwise.

3.3 Computation of the error $(\overline{\mathbf{y}}_{\mathbf{k}}(\mathbf{t}) - \overline{\mathbf{y}}_{\mathbf{k-1}}(\mathbf{t}))$

The (k-1) first derivative of the two outputs coincide at point t=0. So, using Taylor expansion, we can write

$$\overline{y}_k(t) - \overline{y}_{k-1}(t) = \sum_{i \geq k} (\overline{y}_k^{(i)}(0) - \overline{y}_{k-1}^{(i)}(0)).\frac{t^i}{i!}$$

Taking into account that $\overline{y}_k^{(i)}(0) = x_{2k}^{(i)}(0)$, we obtain a right computation of the output's difference at order k and k-1.

4 Numerical examples

Our examples are based on the computation of the two consecutive outputs at order 2 and 3 (we note $E3$ this error), in the case of equations like electric equation. Taylor expansion is at order 5. So we get :

$$\begin{gathered} \overline{y}_3(t) - \overline{y}_2(t) \\ = \\ \frac{t^3.a_2.(a_0 + u(0))^2}{3!} + \\ \frac{t^4(q_{33}^{(4)}(0) - q_{23}^{(4)}(0))}{4!} + \\ \frac{t^5(q_{33}^{(5)}(0) - q_{23}^{(5)}(0))}{5!} + \epsilon(t) \end{gathered}$$

where

$$\epsilon(t) = O(t^6)$$

and

$$(q_{33}^{(4)}(0) - q_{23}^{(4)}(0) =$$
$$2a_1a_2(a_0 + u(0))^2 + 3a_2(a_0 + u(0))u^{(1)}(0)$$

and

$$(q_{33}^{(5)}(0) - q_{23}^{(5)}(0) = 3a_1a_2(a_0 + u(0))^2 +$$
$$7a_1a_2(a_0 + u(0))u^{(1)}(0) +$$
$$a_2^2(a_0 + u(0))^3 +$$
$$4a_2(a_0 + u(0))u^{(2)}(0) +$$
$$3a_2(u^{(1)})^2(0)$$

We want to point up three facts, through these examples :

- The error is too important beyond some interval and our model is not "acceptable"
- The error depends on system behavior and its stability
- The error is different from smooth inputs to rough inputs

5 Conclusion

We have proposed, in this paper, a validation of our model for a particular class of dynamic systems. This validation is not statistical. It consists in valuing the convergence of a bilinear models family (B_k) on the unknown system (Σ) by an effective symbolic computation.
It allows to determine intervals where our model is "'acceptable"', as shown with numerical examples. It displays the respective contributions of the input and of the system itself.
More than a symbolic validation, these computing tools are parameterized by the input and the system's behavior. They can particularly provide a valuation process for rough and oscillating inputs as well as for smooth inputs.

References

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