Maximal replicated submarkets

Ioannis A. Polyrakis * and Foivos Xanthos †

April 28, 2009

Abstract

In this article we study the replication of options in security markets with a finite number of states.

It is known that the space $F_{\mathbf{1}}(X)$, generated by all the possible options written on the elements (marketed securities) of the marketed space X is in general bigger than the space of marketed securities X and usually it is equal to the payoff space $\mathbf{R}^m$ but in many cases $F_{\mathbf{1}}(X)$ is a proper subspace of $\mathbf{R}^m$. If X is a sublattice of $\mathbf{R}^m$ (i.e. X is closed under the operations of supremum and infimum), it is known that $X = F_{\mathbf{1}}(X)$ therefore any option is replicated. But this case is a very rare. In the present article we study the existence of subspaces (submarkets) Y of X so that any option written on the elements of Y to be replicated and also we want Y to be as large as possible. So we study the existence of maximal subspaces of this type which we call maximal replicated submarkets or subspaces. We show that a subspace Y of X is a maximal replicated subspace if the completion $F_{\mathbf{1}}(Y)$ of Y is contained in X and also $Y = F_{\mathbf{1}}(Y)$. In this article we use the theory of lattice-subspaces and positive bases and we find a very easy method by means of which we determine exactly the maximal replicated subspaces of X.

JEL Classification : C600, D520, G190

1 The economic model and some essential notions

In this article we study a two-period security market with a finite number of states $\Omega = \{1, 2, ..., m\}$ during the date 1, a finite number of primitive securities (assets) with payoffs the linearly independent vectors $x_1, x_2, ..., x_n$ of the payoff space $\mathbf{R}^m$.

*Department of Mathematics, National Technical University of Athens, Zographou 157 80, Athens, Greece, e-mail: ypoly@math.ntua.gr

†Department of Mathematics, National Technical University of Athens, Zographou 157 80, Athens, Greece.

A **portfolio** is a vector $\theta = (\theta_1, \theta_2, ...\theta_n)$ of $\mathbb{R}^n$ where θ_i is the number of the units of the i security. Then $T(\theta) = \sum_{i=1}^{n} \theta_i x_i \in \mathbb{R}^m$ is the **payoff of** θ. Since the operator T is one-to-one, it identifies portfolios with their payoffs. So the vectors $x_1, x_2, ..., x_n$ will be mentioned as **primitive securities**, the subspace

$$X = [x_1, x_2, ..., x_n],$$

of E, generated by the vectors x_i as the **space of marketed securities** or the **asset span** and the vectors of X will be also referred as portfolios. We assume that the riskless bond **1** is contained in X. A vector $x \in \mathbf{R}^m$ **is marketed** or x **is replicated** if it is the payoff of some portfolio θ, or equivalently if $x \in X$.
Recall that the vector space $\mathbf{R}^m = \{x = (x(1), x(2), ..., x(m)) | x(i) \in \mathbf{R} \text{ for each } i\}$ is ordered by the pointwise ordering i.e. for any $x, y \in \mathbf{R}^m$ we have: $x \geq y$ if $x(i) \geq y(i)$ for each i. $\mathbf{R}^m_+ = \{x \in \mathbf{R}^m | x(i) \geq 0 \text{ for each } i\}$ is the positive cone of $\mathbf{R}^m$. For any $x, y \in \mathbf{R}^m$ $x \vee y = \big(x(1) \vee y(1), x(2) \vee y(2), ..., x(m) \vee y(m)\big)$ is the supremum and $x \wedge y = \big(x(1) \wedge y(1), x(2) \wedge y(2), ..., x(m) \wedge y(m)\big)$ is the infimum of $\{x, y\}$ in $\mathbf{R}^m$. $x^+ = x \vee 0 = \big(x(1) \vee 0, x(2) \vee 0, ..., x(m) \vee 0\big)$ and $x^- = (-x) \vee 0$ are the positive and the negative part of x. Note also that for any two real numbers a, b, $a \vee b$ is the supremum and $a \wedge b$ is the infimum of $\{a, b\}$. A linear subspace Z of $\mathbf{R}^m$ is a **sublattice** or a **Riesz subspace** of $\mathbf{R}^m$ if for any $x, y \in Z$, $x \vee y$ and $x \wedge y$ belong to Z. Also for any $x = (x(1), x(2), ..., x(m)) \in \mathbf{R}^m$, the set $supp(x) = \{i = 1, 2, ..., m | x(i) \neq 0\}$ is the support of x.
For any subset B of $\mathbf{R}^m$, the **sublattice** $S(B)$ **of** $\mathbf{R}^m$ **generated by** B is the intersection of the sublattices of $\mathbf{R}^m$ which contain B. The **riskless bond 1** is the vector of $\mathbb{R}^m$ whose every coordinate is equal to 1. The **call option** written on the vector $x \in \mathbb{R}^m$ with exercise price a is the vector $c(x, a) = (x - a\mathbf{1})^+$ of $\mathbb{R}^m$ and the **put option** of x with exercise price a is $p(x, a) = (a\mathbf{1} - x)^+$. We have $x - a\mathbf{1} = c(x, a) - p(x, a)$.
If both $c(x, a) > 0$ and $p(x, a) > 0$, we say that the call option $c(x, a)$ is **non trivial** and if $c(x, a), p(x, a)$ belong to X we say that $c(x, a), p(x, a)$ are w **replicated**. The **completion by options** of X is the subspace of $\mathbb{R}^m$ which arises inductively by adding in the market the call and put options of the marketed securities and by taking again call and put options which are added again in the market. In [3] a mathematical definition of the completion by options in infinite securities markets is given. Especially in the above article, a more general study of the completion by options of the market is presented were the options are not taken with respect to the riskless bond **1** but with respect to some risky vectors from a standard subspace U of $\mathbb{R}^m$ and the completion by options of X is denoted by $F_U(X)$. This study is very general and includes the case of exotic options. In the classical case where the options are taken with respect to the riskless bond **1**, the completion by options

of X is denoted in [3] by $F_{\mathbf{1}}(X)$ and we will preserve this notation in the present article. In [3] it is proved that if the payoff space is a general vector lattice E then $F_U(X)$ is the sublattice of E generated by the set $X \cup U$. In our case where the payoff space is the space $\mathbb{R}^m$ and the call and put options are taken with respect to the riskless bond $\mathbf{1}$, the completion by options $F_{\mathbf{1}}(X)$ of X is the sublattice of $\mathbb{R}^m$ generated by the set $X \cup \{\mathbf{1}\}$. In the case where $\mathbf{1} \in X$, $F_{\mathbf{1}}(X)$ is the sublattice of $\mathbb{R}^m$ generated by the set X. In the above results and also in the results of the present article, the theory of lattice-subspaces and positive bases developed by Polyrakis in the papers of the bibliography is very important. This theory simplifies and unifies the theory of options.
In the present article we study the existence of subspaces Y of the marketed space X so that any option written on the elements of Y to be replicated. In this case Y is a submarket so that any option written on a portfolio of securities of Y to be replicated by a portfolio of marketed securities. We want also this market to be as larger as possible so we are looking for maximal replicated submarkets (subspaces of X). As we have noted in the introduction, we develop a method which determines the maximal replicated subspaces Y of X. Especially we determine a positive basis $\{d_i\}$ of Y. The elements d_i of this basis are securities which generate the portfolios in Y. In [5], the next characterization of markets without binary vectors is proved:
X does not contain binary vectors if and only if for any non-constant vector $x \in X$ (i.e. for any x which is not a multiple of $\mathbf{1}$) at least one non-trivial option of x is non-replicated.
So is X contains binary vectors, the only replicated subspace of X is the one dimensional subspace generated by the riskless bond $\mathbf{1}$. Also in the case where X is a sublattice of $\mathbf{R}^m$ then any option is replicated, therefore $F_{\mathbf{1}}(X) = X$ and the whole market X is replicated.
In the case where the market X contains binary vectors and also X is not a sublattice of $\mathbf{R}^m$, there exist non-trivial maximal replicated markets and this is the case we study in the present article.
For a study of the two-periods security markets we refer to the book of LeRoy and Werner (2001), [4]. We refer also to the articles of Ross [9], Aliprantis-Tourky [1] and Baptista [2].

2 The completion by options $F_{\mathbf{1}}(X)$ of X

In this section we describe the method of determination of the completion by options of X, which is denoted by $F_{\mathbf{1}}(X)$, as it is presented in [3]. According to this method we consider the set

$$\mathcal{A} = \{x_1^+, x_1^-, x_2^+, x_2^-, ..., x_n^+, x_n^-, \mathbf{1}\}.$$

Any maximal subset $\{y_1, y_2, ..., y_r\}$ of linearly independent vectors of $\mathcal{A}$ is a **basic set** of the market, where x_i^+, x_i^- are the positive and negative parts of the vectors x_i. Note that a basic set is not necessarily unique. In general it is possible to find different basic sets of the market but all these sets have the same cardinal number r. Especially r is the dimension of the linear subspace of $\mathbf{R}^m$ generated by $\mathcal{A}$ and a basic set is a basis of it.

Theorem 1 ([3], Theorem11). *$F_{\mathbf{1}}(X)$ is the sublattice of $\mathbf{R}^m$ generated by a basic set $\{y_1, y_2, \dots, y_r\}$ of the market.*

After this result we use the theory of lattice-subspaces and positive bases developed by Polyrakis in [6] and [7] for the determination of $F_{\mathbf{1}}(X)$. Since $F_{\mathbf{1}}(X)$ is a sublattice of $\mathbf{R}^m$ which contains $\mathbf{1}$, we have that $F_{\mathbf{1}}(X)$ has a positive basis $\{b_1, b_2, ..., b_\mu\}$ which is a **partition of the unit**, i.e. the vectors b_i have disjoint supports and $\sum_{i=1}^{\mu} b_i = \mathbf{1}$, see in Theorem 14 of the Appendix. This basis is unique. So we have:

Theorem 2. *$F_{\mathbf{1}}(X)$ has a positive basis $\{b_1, b_2, ..., b_\mu\}$ which is a partition of the unit.*

For the determination of the positive basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$ which is a partition of the unit we follow the steps of Polyrakis algorithm, see Theorem 16 in the appendix, where a positive basis of the sublattice of $\mathbf{R}^m$ generated by a finite set of positive and linearly independent vectors is determined. We start by the determination of a basic function set of $\{y_1, y_2, \dots, y_r\}$ of the market. In the sequel we determine the **basic function** of $y_1, y_2, \dots, y_r$ which is very important for the theory of lattice-subspaces and positive bases. This function has been defined in [6] and is the following:

$$\beta(i) = \Big(\frac{y_1(i)}{y(i)}, \frac{y_2(i)}{y(i)}, ..., \frac{y_r(i)}{y(i)}\Big), \text{ for each } i = 1, 2, ..., m, \text{ with } y(i) > 0,$$

where $y = y_1 + y_2 + ... + y_r$. This function takes values in the simplex $\Delta_r = \{\xi \in \mathbb{R}_+^r | \sum_{i=1}^{r} \xi_i = 1\}$ of $\mathbb{R}_+^r$.
Denote by $R(\beta)$ is the range (i.e. the set of values) of β and by $cardR(\beta)$ the cardinal number of $R(\beta)$ (i.e. the number of the different values of β).
We continue the algorithm and we obtain a positive basis $\{d_1, d_2, ..., d_\mu\}$ of $F_{\mathbf{1}}(X)$. The elements of this basis have disjoint support and each b_i is constant on its support. So by a normalization of the basis $\{d_i\}$ we obtain the basis positive basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$ which is also a partition of the unit.

Except the determination of the positive basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$, by the basic function we can ask very easy to two important questions of the theory of options. Especially we can check directly wether any option is replicated and also if the completion by options of X is the whole space $\mathbb{R}^m$. Recall that if $X = F_{\mathbf{1}}(X)$, any option is replicated and we say then that X is **complete by options** (with respect to $\mathbf{1}$). As it is proved in Polyrakis (1999), Theorem 3.7 (see also in the Appendix) the dimension of the sublattice generated by $\{y_1, y_2, \ldots, y_r\}$, i.e. the dimension of $F_{\mathbf{1}}(X)$, is equal to the cardinal number of $R(\beta)$. So if $R(\beta)$ has n elements we have that $F_{\mathbf{1}}(X) = X$ and if $R(\beta)$ has m elements we have that $F_{\mathbf{1}}(X) = \mathbb{R}^m$. so we have:

Theorem 3. *The dimension of $F_{\mathbf{1}}(X)$ is equal to the cardinal number of the range $R(\beta)$, therefore we have:*

(i) $F_{\mathbf{1}}(X) = X$ *if and only if* $cardR(\beta) = n$,

(ii) $F_{\mathbf{1}}(X) = \mathbb{R}^m$ *if and only if* $cardR(\beta) = m$,

(iii) $F_{\mathbf{1}}(X) \subsetneq \mathbb{R}^m$ *if and only if* $cardR(\beta) < m$.

3 Replication and projection bases

The notion of the projection basis has been defined by Polyrakis, [8] for a finite dimensional subspace L of $C(\Omega)$ generated by the linearly independent positive vectors $y_1, y_2, ..., y_r$ of $C(\Omega)$ which is contained (the subspace L) in a finite dimensional lattice-subspace W of $C(\Omega)$. This basis, $\{\widetilde{b_1}, \widetilde{b_2}, ..., \widetilde{b_r}\}$, is called projection basis because its elements are projections of the elements of the positive basis of W. In the present paper we construct a projection basis basis of L, in the case where the sublattice Z of $C(\Omega)$ generated by L is finite dimensional, by taking the projections of the positive basis of Z. The philosophy of this result and also the proof are the same with the ones in [8], Theorem 9. We give this result below without proof and we do not consider it as new.

Theorem 4 (Polyrakis). *Let Z be the sublattice of $C(\Omega)$ generated by the linearly independent and positive vectors $z_1, z_2, ..., z_r$ of $C(\Omega)$ and suppose that $dim(Z) = \mu$. Suppose that we follow the steeps of statement (ii) (and the same notions) of Theorem 16 in the Appendix for the determination of a positive basis $\{b_1, b_2, \ldots, b_\mu\}$ of Z which is given by (5). So we suppose that β is the basic function of the vectors z_i, $P_1, P_2, \ldots, P_r, P_{r+1}, \ldots, P_\mu$ is the enumeration of the range of β of Theorem 16 so that the first r vectors $P_1, P_2, ..., P_r$ to be linearly independent and suppose also that $z_{r+1}, z_{r+2}, ..., z_\mu$ are the new vectors constructed in (b) of the Theorem.*

If $L = [z_1, z_2, \ldots, z_r]$ is the subspace of $C(\Omega)$ generated by the vectors $z_1, z_2, ..., z_r$ we have:

(i) $Z = L \oplus [z_{r+1}, z_{r+2}, ..., z_\mu]$.

(ii) $b_i = 2z_i$, *for each* $i = r+1, r+2, ..., \mu$.

(iii) If $b_i = \widetilde{b_i} + b'_i$, *with* $\widetilde{b_i} \in L$ *and* $b'_i \in [z_{r+1}, z_{r+2}, ..., z_\mu]$, *for each* $i = 1, 2, ..., r$, *then* $\{\widetilde{b_1}, \widetilde{b_2}, ..., \widetilde{b_r}\}$ *is a basis of* L *which we call* **projection basis** *of* L *and is given by the formula*

$$(\widetilde{b_1}, \widetilde{b_2}, ..., \widetilde{b_r})^T = A^{-1}(z_1, z_2, ..., z_r)^T, \tag{1}$$

where A *is the* $n \times n$ *matrix whose the* i^{th} *column is the vector* P_i, *for* $i = 1, 2, ..., r$. *This basis has the property: The* r *first coordinates of any element* x *of* L *in the basis* $\{b_1, b_2, ..., b_\mu\}$ *coincide with the corresponding coordinates of* x *in the projection basis, i.e.*

$$x = \sum_{i=1}^{\mu} \lambda_i b_i \in L \implies x = \sum_{i=1}^{r} \lambda_i \widetilde{b_i}. \tag{2}$$

The proof of the next proposition is very easy.

Proposition 5. *Suppose that* b_i *is the basis of* $\boldsymbol{F_1}(X)$, $\widetilde{b_i}$ *is the projection basis of* L *defined in the previous theorem and suppose that* $\mathbf{1} = \sum_{i=1}^{\mu} \rho_i b_i$ *is the expansion of* $\mathbf{1}$. *Then* $\{d_i = \rho_i b_i | i = 1, 2, ..., \mu\}$ *is a positive basis of* Z *which is a partition of the unit and* $\{\widetilde{d_i} = \rho_i \widetilde{b_i} | i = 1, 2, ..., r\}$ *is a positive basis of* L, *which we call* **projection basis of** L **corresponding to the basis** $\{d_i\}$ *of* Z *and for any* $x \in L$ *we have:*

$$x = \sum_{i=1}^{\mu} \lambda_i b_i \in L \implies x = \sum_{i=1}^{r} \lambda_i \widetilde{b_i}. \tag{3}$$

Remark 6. From the construction of a projection basis of Theorem 16 we have that in general, it is possible to found different projection basis on L. Indeed, in the above algorithm in the point we take a new enumeration of $\{P_1, P_2, ..., P_\mu\}$ so that the r first P_i to be linear independent we have many possibilities. Not that we can find always r linear independent vectors of P_i. So if d is the number of all such the choices we have: $1 \leq d \leq \binom{\mu}{r}$.

Definition 7. *Suppose that* Y *is a subspace of* X. *If the completion by options of* Y *is contained in* X, *i.e.* $F_\mathbf{1}(Y) \subseteq X$, *we say that* Y *is* **replicated** *and if moreover for any subspace* Z *of* X *with* $Y \subsetneqq Z$ *we have* $X \subsetneqq F_\mathbf{1}(Z)$, *we say that* Y *is a* **maximal replicated subspace** *or* **maximal replicated submarket** *of* X.

Proposition 8. *Suppose that Y is a maximal replicated subspace of X. Then Y is a maximal sublattice of $\mathbb{R}^m$ contained in X with $Y = F_{\mathbf{1}}(Y)$, or equivalently Y is a maximal sublattice of $\mathbb{R}^m$ contained in X with $\mathbf{1} \in Y$.*

Proof. Y is a replicated subspace of X, therefore $Y \subseteq F_{\mathbf{1}}(Y) \subseteq X$. Suppose that Z is a subspace of $F_{\mathbf{1}}(Y)$ with $Y \subsetneqq Z$. Then $\mathbf{1} \in F_{\mathbf{1}}(Y)$. Also the completion $F_{\mathbf{1}}(Z)$ of of Z is the sublattice generated by $Z \cup \{\mathbf{1}\}$, therefore $F_{\mathbf{1}}(Z) \subseteq F_{\mathbf{1}}(Y) \subseteq X$, so Z is replicated. This is a contradiction because Y is a maximal replicated subspace of X, therefore $Y = F_{\mathbf{1}}(Y)$. So Y is a sublattice $\mathbb{R}^m$. To show that Y is maximal suppose that $F_{\mathbf{1}}(Y) \subsetneqq Z \subseteq X$ and that Z is a sublattice. Then $\mathbf{1} \in Z$, therefore we replicated. Then $Z = F_{\mathbf{1}}(Z) \subseteq X$, so Z is replicated. This is a contradiction therefore Y is a maximal sublattice of $\mathbb{R}^m$ contained in X. By the above proof it is also clear that Y is a maximal sublattice of $\mathbb{R}^m$ contained in X with $\mathbf{1} \in Y$. ■

Definition 9. Let $\{b_i, i = 1, ..., \mu\}$ the positive basis of $\mathbf{F_1}(X)$ which is a partition of the unit and let $\{\widetilde{b_i} |\ i = 1, ..., n\}$ be the corresponding projection basis of X. A partition $\delta = \{\sigma_i |\ i = 1, ..., \kappa\}$ of $\{1, ..., n\}$ is **proper** if for any $r = 1, 2, ..., \kappa$, the vector $w_r = \Sigma_{i \in \sigma_r} \widetilde{b_i}$ is a binary vector (i.e. $w_r(i) = 1$ or $w_r(i) = 0$ for any $i = 1, 2, ..., m$) with $\Sigma_{r=1}^{\kappa} w_r = \mathbf{1}$. If moreover δ is maximal, in the sense that does not exists a partition φ of $\{1, ..., n\}$ finer than δ (the partition $\varphi = \{\omega_i |\ i = 1, ..., \tau\}$ is finer than δ if any ω_i is a subset of some σ_j and the inclusion is proper for at least one ω_i), then we say that δ is a **maximal proper** partition of $\{1, ..., n\}$.

Theorem 10. *Let $\{b_i, i = 1, ..., \mu\}$ the positive basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit and let $\{\widetilde{b_i}, i = 1, ..., n\}$ be the corresponding projection basis of X. Then X has a maximal replicated subspace Y if and only if there exists a maximal proper partition $\delta = \{\sigma_i |\ i = 1, ..., \kappa\}$ of $\{1, ..., n\}$.*
If $\delta = \{\sigma_i |\ i = 1, ..., \kappa\}$ is a maximal proper partition of $\{1, ..., n\}$ and if $d_r = \sum_{i \in \sigma_r} \widetilde{b_i}$ for any $r = 1, ..., \kappa$, then the subspace Y of X generated by the vectors $d_1, d_2, ...d_\kappa$ is a maximal a sublattice of $\mathbb{R}^m$ contained in X and $\{d_1, d_2, ...d_\kappa\}$ is a positive basis of Y which is a partition of the unit.
If Y is a maximal replicated subspace of X, then Y is is independent on the choice of the basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$ and the projection basis $\{\widetilde{b_i}\}$ of X.

Proof. Suppose that Y is a maximal replicated subspace of X. Then Y is a maximal sublattice of $\mathbb{R}^m$ contained in X with $Y = F_{\mathbf{1}}(Y)$. Also $\mathbf{1} \in Y$. So Y has a positive basis $\{d_1, d_2, ..., d_k\}$ which is a partition of the unit. Therefore $supp(d_i) \cap supp(d_j) = \emptyset$ for any $i \neq j$ and $\sum_{r=1}^{k} d_r = \mathbf{1}$. Suppose that $d_r = \sum_{i \in \Phi_r} b_i$ where $\Phi_r \subseteq \{1, 2, ..., \mu\}$, is the expansion of d_r in the basis $\{b_i\}$. Since the basis $\{b_i\}$ is a partition of the unit, it is easy to show that

$\{\Phi_r|\ r = 1,2,...,k\}$ is a partition of $\{1,2,...,\mu\}$. Since $d_r \in X$, by Theorem **??**, we have that $d_r = \sum_{i\in\sigma_r} \widetilde{b_i}$, where $\sigma_r = \Phi_r \cap \{1,2,...,\mu\}$. Since $\sum_{i=1}^{\mu} b_i = \sum_{r=1}^{k} d_r = \mathbf{1}$ we have that $\sigma_r \neq \emptyset$ for any $r = 1,2,...,k$.
It is easy to see that $\delta = \{\sigma_r | r = 1,2,...,k\}$ is a proper partition of $\{1,2,...,n\}$. Also δ is maximal because if we suppose that $\varphi = \{\omega_i|\ i = 1,...,\tau\}$ is a partition of $\{1,...,n\}$ finer than δ we have: If for any $r = 1,...,\tau$, we put $z_r = \sum_{i\in\omega_r} \widetilde{b_i}$ and Z is the subspace of $\mathbb{R}^m$ generated by the vectors $z_r | r = 1,2,...,\tau$, then $Z \subseteq X$ is a sublattice which contains $\mathbf{1}$ and also $Y \subsetneqq Z$. The last relation holds by our assumption that ω is finer than δ. This is a contradiction, therefore δ is maximal.
For the converse suppose that $\delta = \{\sigma_i|\ i = 1,...,\kappa\}$ is a maximal proper partition of $\{1,...,n\}$. For any $r = 1,...,\kappa$, we put $z_r = \sum_{i\in\sigma_r} \widetilde{b_i}$ and suppose that Y is the subspace of $\mathbb{R}^m$ generated by the vectors $b_r,\ r = 1,2,...,\kappa$. Then Y is a sublattice of $\mathbb{R}^m$ which is contained in X. Also $\mathbf{1} \in Y$ and Y is a maximal sublattice which is contained in X because if we suppose that $Z \subseteq X$ is a sublattice which contains Y as a proper subspace by the positive basis of Z we get a proper decomposition of $\{1,...,n\}$ finer than δ. This is a contradiction, therefore Y is a maximal replicated subspace of X.
Suppose that Y is a maximal replicated subspace which is determined as in the first part of our proof. To show that Y independent on the choice of the basis $\{b_i\}$ of $F_{\mathbf{1}}(X)$ and the projection basis $\{\widetilde{b_i}\}$ of X we return back in the first part of the proof. We remark that the basis of $F_{\mathbf{1}}(X)$ which is a partition of the unit is unique, so any such a basis $\{b'_i\}$ of $F_{\mathbf{1}}(X)$ is at most a new enumeration of $\{b_i\}$. So we suppose that $\{b_{i_1}, b_{i_2}, ..., b_{i_\mu}\}$ is a basis of $F_{\mathbf{1}}(X)$. Since the expansion of d_r is unique, we have that $d_r = \sum_{\nu\in\Phi'_r} b_{i_\nu}$, where $\nu \in \Phi'_r$ if and only if $i_\nu \in \Phi_r$. Then $\{\Phi'_r|\ r = 1,2,...,k\}$ is a partition of $\{1,2,...,\mu\}$ and $\sigma'_r = \Phi'_r \cap \{1,2,...,n\}$. Then $\{\sigma'_r|\ r = 1,2,...,k\}$ is a partition of $\{1,2,...,n\}$ and according to the first part of our proof, the maximal replicated subspace Y' generated by this partition of $\{1,2,...,n\}$ has as positive basis the basis $\{d'_r = \sum_{\nu\in\Phi'_r} b_{i_\nu}\}$. It is easy to see that $d'_r = d_r$, for any r, therefore $Y' = Y$.

■

The next is an example of the determination of the maximal replicated subspaces. As we have remarked before, we can determine different projection bases of X. According to our method we determine the range $R(\beta)$ of the basic function β and we enumerate it so that its first n vectors to be linearly independent. For any such enumeration in general we find a different projection basis. In the next example we have four different enumerations of $R(\beta)$ which gives three different projection bases but the same maximal replicated subspace.

Example 11. Suppose that $x_1 = (2,2,4,1,2),\ x_2 = (0,1,1,0,0), x_3 = (1,1,1,1,1)$

are the primitive securities and $X=[x_1,x_2,x_3]$ is the marketed space.
According to the methodology of the determination of $F_{\mathbf{1}}(X)$, a basic set of the market is $\{x_1,x_2,x_3\}$ and $\beta=\frac{1}{x}(x_1,x_2,x_3)$ where x is the sum of x_i, is the basic function of the vectors $x-i$. We find that

$$\beta(1)=\beta(5)=\frac{1}{3}(2,0,1)=P_1,\beta(2)=\frac{1}{4}(2,1,1)=P_2,$$

$$\beta(3)=\frac{1}{6}(4,1,1)=P_3,\beta(4)=\frac{1}{2}(1,0,1)=P_4.$$

In order to determine a positive basis of $F_{\mathbf{1}}(X)$ we start by an enumeration of the range $R(\beta)$ of β so that the first three vectors of $R(\beta)$ to be linearly independent. So we have the following different enumerations:
(i):

$$R(\beta)=\{P_1,P_2,P_3,P_4\}.$$

According to our methodology, $I_4=\beta^{-1}(P_4)=\{4\}$ and we define the new vector $x_4=(0,0,0,2,0)$. $\gamma=\frac{1}{x'}(x_1,x_2,x_3,x_4)$ where x' is the sum of the vectors x_i, is the basic function of x_1,x_2,x_3,x_4. We have

$$\gamma(1)=\gamma(5)=\frac{1}{3}(2,0,1,0)=P'_1,\gamma(2)=\frac{1}{4}(2,1,1,0)=P'_2$$

$$\gamma(3)=\frac{1}{6}(4,1,1,0)=P'_3,\gamma(4)=\frac{1}{4}(1,0,1,2)=P'_4.$$

A positive basis of $F_{\mathbf{1}}(X)$ is given by the formula $(d_1,d_2,d_3,d_4)^T=D^{-1}(x_1,x_2,x_3,x_4)^T$ where D is the matrix whose columns are the vectors $P'_i, i=1,2,3,4$ and we find that the vectors

$$d_1=(3,0,0,0,3),d_2=(0,4,0,0,0),d_3=(0,0,6,0,0),d_4=(0,0,0,4,0),$$

define a positive basis of $\mathbf{F_1}(X)$. A projection basis of X is given by the formula $(\tilde{d}_1,\tilde{d}_2,\tilde{d}_3)^T=A^{-1}(x_1,x_2,x_3,x_4,x_5)^T$ where A is the matrix with columns are the vectors P_1,P_2,P_3 and we find that

$$\tilde{d}_1=(3,0,0,3,3),\tilde{d}_2=(0,4,0,2,0),\tilde{d}_3=(0,0,6,-3,0)$$

is a projection basis of X. The positive basis of $\mathbf{F_1}(X)$ which is a partition of the unite is

$$b_1=(1,0,0,0,1),b_2=(0,1,0,0,0),b_3=(0,0,1,0,0)$$

$$b_4 = (0,0,0,1,0),$$

and the corresponding projection basis of X is

$$\tilde{b}_1 = (1,0,0,1,1), \tilde{b}_2 = (0,1,0,\frac{1}{2},0), \tilde{b}_3 = (0,0,1,-\frac{1}{2},0).$$

We remark that $q_1 = \tilde{b}_1 = (1,0,0,1,1)$ and $q_2 = \tilde{b}_2 + \tilde{b}_3 = (0,1,1,0,0)$ and it is easy one to see that $\{1\},\{2,3\}$ is a maximal proper decomposition of $\{1,2,3\}$. So $Y = [(1,0,0,1,1),(0,1,1,0,0)]$ is a maximal replicated subspace of the market.
(ii) If we start by the enumeration If

$$R(\beta) = \{P_1, P_3, P_4, P_2\},$$

by similarly we find the projection basis

$$\tilde{b}_1 = (1,-2,0,0,1), \tilde{b}_2 = (0,1,1,0,0), \tilde{b}_3 = (0,2,0,1,0)$$

of X. So we have that $q_1 = \tilde{b}_2 = (0,1,1,0,0)$, $q_2 = \tilde{b}_1 + \tilde{b}_3 = (1,0,0,1,1)$, therefore $\{2\},\{1,3\}$ is a maximal proper decomposition of $\{1,2,3\}$ and we find again the same maximal replicated subspace Y of the market.
(iii) If

$$R(\beta) = \{P_2, P_3, P_4, P_1\},$$

the projection basis of X is

$$\tilde{b}_1 = (-\frac{1}{2},1,0,0,-\frac{1}{2}), \tilde{b}_2 = (\frac{1}{2},0,1,0,\frac{1}{2}), \tilde{b}_3 = (1,0,0,1,1)$$

and we see that $q_1 = \tilde{b}_3 = (1,0,0,1,1)$ and $q_2 = \tilde{b}_1 + \tilde{b}_2 = (0,1,1,0,0)$, therefore $\{3\},\{1,2\}$ is a maximal proper decomposition of $\{1,2,3\}$ which gives the same maximal replicated subspace Y.
(iv): If

$$R(\beta) = \{P_1, P_2, P_4, P_3\}$$

we find that the projection basis of case (i) and the same maximal replicated subspace.

Example 12. Suppose that $x_1 = (1,2,3,4,5,6)$, $x_2 = (2,0,1,0,0,1), x_3 = (1,1,1,1,1,1), x_4 = (2,1,1,3,0,0), x_5 = (0,0,0,0,5,4)$ are the primitive securities and $X = [x_1, x_2, x_3, x_4, x_5]$ is the marketed space.

According to the methodology of the determination of $F_{\mathbf{1}}(X)$ we have that a basic set of the market is $\{x_1, x_2, x_3, x_4, x_5\}$. The basic function is $\beta = \frac{1}{x}(x_1, x_2, x_3, x_4, x_5)$ where x is the sum of x_i and we find that

$$\beta(1) = \frac{1}{6}(1,2,1,2,0) = P_1, \beta(2) = \frac{1}{4}(2,0,1,1,0) = P_2, \beta(3) = \frac{1}{6}(3,1,1,1,0) = P_3$$

$$\beta(4) = \frac{1}{8}(4,0,1,3,0) = P_4, \beta(5) = \frac{1}{11}(5,0,1,0,5) = P_5, \beta(6) = \frac{1}{12}(6,1,1,0,4) = P_6.$$

So we have that $card(R(\beta)) = 6$ therefore the completion $\mathbf{F_1}(X)$ is $\mathbf{R}^6$. The five first vectors P_1, P_2, P_3, P_4, P_5 of $R(\beta)$ are linearly independent,so we preserve the enumeration of $R(\beta)$.Acording to our methodology $I_6 = \beta^{-1}(P_6) = \{6\}$ and we define the new vector $x_6 = (0,0,0,0,0,12)$. We determine the basic function $\gamma = \frac{1}{x'}(x_1, x_2, x_3, x_4, x_5, x_6)$ where x' is the sum of these vectors. We find that

$$\gamma(1) = \frac{1}{6}(1,2,1,2,0,0) = P'_1, \gamma(2) = \frac{1}{4}(2,0,1,1,0,0) = P'_2, \gamma(3) = \frac{1}{6}(3,1,1,1,0,0) = P'_3$$

$$\gamma(4) = \frac{1}{8}(4,0,1,3,0,0,0) = P'_4, \gamma(5) = \frac{1}{11}(5,0,1,0,5,0) = P'_5, \gamma(6) = \frac{1}{24}(6,1,1,0,4,12) = P'_6.$$

A positive basis of $F_{\mathbf{1}}(X)$ is given by the formula $(d_1, d_2, d_3, d_4, d_5, d_6)^T = D^{-1}(x_1, x_2, x_3, x_4, x_5, x_6)^T$ where D is the matrix whose columns are the vectors $P'_i, i = 1, ..., 6$ and we find that the vectors

$$d_1 = (6,0,0,0,0,0), d_2 = (0,4,0,0,0,0), d_3 = (0,0,6,0,0,0)$$

$$d_4 = (0,0,0,8,0,0), d_5 = (0,0,0,0,11,0), d_6 = (0,0,0,0,0,24)$$

define a positive basis of $\mathbf{F_1}(X)$. The projection basis of X with respect to the above positive basis of $\mathbf{F_1}(X)$ is given by the formula $(\tilde{d}_1, \tilde{d}_2, \tilde{d}_3, \tilde{d}_4, \tilde{d}_5)^T = A^{-1}(x_1, x_2, x_3, x_4, x_5)^T$ where A is the matrix whose columns are the vectors $P_i, i = 1, ..., 5$ and we find that the following vectors form these projection basis.

$$\tilde{d}_1 = (6,0,0,0,0,-1.2), \tilde{d}_2 = (0,4,0,0,0,-4), \tilde{d}_3 = (0,0,6,0,0,8.4)$$

$$\tilde{d}_4 = (0,0,0,8,0,0), \tilde{d}_5 = (0,0,0,0,11,8.8)$$

We can easily get the proper form of the above basis which is the following:

$$b_1 = \frac{1}{6}d_1 = (1,0,0,0,0,0), b_2 = \frac{1}{4}d_2 = (0,1,0,0,0,0), b_3 = \frac{1}{6}d_3 = (0,0,1,0,0,0)$$

$$b_4 = \frac{1}{8}d_4 = (0,0,0,1,0,0), b_5 = \frac{1}{11}d_5 = (0,0,0,0,1,0), b_6 = \frac{1}{24}d_6 = (0,0,0,0,0,1).$$

The corresponding projection basis is

$$\tilde{b}_1 = \frac{1}{6}\tilde{d}_1 = (1,0,0,0,0,-0.2), \tilde{b}_2 = \frac{1}{4}\tilde{d}_2 = (0,1,0,0,0,-1), \tilde{b}_3 = \frac{1}{6}\tilde{d}_3 = (0,0,1,0,0,1.4)$$

$$\tilde{b}_4 = \frac{1}{8}\tilde{d}_4 = (0,0,0,1,0,0), \tilde{b}_5 = \frac{1}{11}\tilde{d}_5 = (0,0,0,0,1,0.8).$$

Then using the above method we see that $q_1 = \tilde{b}_4 = (0,0,0,1,0,0)$ and $q_2 = \tilde{b}_1 + \tilde{b}_2 + \tilde{b}_3 + \tilde{b}_5 = (1,1,1,0,1,1)$, therefore $\{1,2,3,5\}, \{4\}$ is a maximal proper decomposition of $\{1,2,3,4,5\}$. So Y generated by q_1, q_2 is a maximal replicated subspace of the market.

4 Appendix: Sublattices and positive bases in $C(\Omega)$

A real vector space E is an *ordered vector space* if it is endowed with a partial order (i.e reflexive, antisymmetric and transitive) relation $\geq$ which is compatible with the linear structure of E, i.e. $x \geq y$ implies $x + z \geq y + z$ and $ax \geq ay$, for any $z \in E$ and $a \in \mathbb{R}_+$. The set $E_+ = \{x \in E | x \geq 0\}$ is *positive cone* of E. E is a *Riesz space* or a *vector lattice* if for any $x, y \in E$ the supremum $x \vee y$ and the infimum $x \wedge y$ of the set $\{x, y\}$ in E exists. A subspace Z of E ordered by the induced ordering is an ordered subspace of E. Then $Z_+ = Z \cap E_+$ is is the positive cone of Z. If moreover for any $x, y \in Z$ the supremum $x \vee_Z y$ and the infimum $x \wedge_Z y$ of $\{x, y\}$ exist in Z (i.e. Z is a Riesz space in the induced ordering), Z is a *lattice-subspace*of E. For any $x \in E$ where E is a Riesz space the element $x^+ = x \vee 0$ is the *positive part* of x and the element $x^- = (-x) \vee 0$ is the *negative part* of x. For any $x \in E$, $x = x^+ - x^-$ holds. If $x \vee_Z y = x \vee y, x \wedge y = x \wedge_Z y$ for every $x, y \in Z$, then Z is a *sublattice* of E. If E is a Riesz space and B is a nonempty set of E_+ then the *minimum sublattice* of E containing B exists and it is the subspace of E generated by the set of finite suprema of elements of B. Let L be the set of lattice -subspaces of E which contain B. X is a *minimal lattice-subspace* of E containing B if $X \in L$ and for any $Y \in L$ such that $Y \subseteq X$, $Y = X$ holds.

If Z is a finite-dimensional subspace of the Riesz space E then $\{b_1, b_2, ..., b_r\} \subseteq Z$ is a *positive basis* of Z if it is a basis of Z and $Z_+ = \{x = \sum_{i=1}^{r} \lambda_i b_i | \lambda_i \in \mathbb{R}_+$ for any $i\}$. If $\{b_1, b_2, ..., b_r\}$ is a positive basis of Z and $x = \sum_{i=1}^{r} \lambda_i b_i, y = \sum_{i=1}^{r} \mu_i b_i$ with $x \geq y$, then $\lambda_i \geq \mu_i$ for any $i = 1, 2, ..., r$. Moreover, in this case $x \vee_Z y = \sum_{i=1}^{r} (\lambda_i \vee \mu_i) b_i$ and $x \wedge_Z y = \sum_{i=1}^{r} (\lambda_i \wedge \mu_i) b_i$.
Suppose that $E = C(\Omega)$ where Ω is a Hausdorff, compact topological space and the order of $C(\Omega)$ is actually the pointwise order (i.e the positive cone of $C(\Omega)$ with respect to this order is $C_+(\Omega) = \{x \in C(\Omega) | x(t) \geq 0$, for any $t \in \Omega\}$).
Suppose that $C(\Omega)$ is the space of real valued functions defined on a compact Hausdorff topological space Ω. Recall that if the set Ω is finite, for example $\Omega = \{1, 2, ..., m\}$, then any element of $C(\Omega)$ is a vector of $\mathbb{R}^m$ therefore $C(\Omega)$ is the space $\mathbb{R}^m$.
In this section we give some basic mathematical notions and results in $C(\Omega)$ which are needed for this article. $C(\Omega)$ is ordered by the pointwise ordering i.e. for any $x, y \in C(\Omega)$ we have $x \geq y$ if and only if $x(i) \geq y(i)$ for each $i \in \Omega$. $C(\Omega)_+ = \{x \in C(\Omega) \big| x(i) \geq 0$ for each $t \in \Omega\}$ is the positive cone of $C(\Omega)$. Suppose that L is an **ordered subspace** of $C(\Omega)$, i.e. L is a subspace of $C(\Omega)$ ordered again by the pointwise ordering. Then $L_+ = C(\Omega)_+ \cap L$ is the positive cone of L. Suppose that L is finite dimensional. A basis $\{b_1, b_2, ..., b_r\}$ of L is a **positive basis** of L if $L_+ = \{x = \sum_{i=1}^{r} \lambda_i b_i \mid \lambda_i \in \mathbb{R}_+$ for each $i\}$. In other words, a basis of L is positive if for any $x \in L$ we have: x is positive if and only its coefficients in the basis are positive. Although L has infinitely many bases the existence of a positive basis of L is not always ensured. We have: L has a positive basis if and only if L is a **lattice-subspace** of $\mathbb{R}^m$ (i.e. if L in the pointwise ordering is a vector lattice). As we have noted in the previous section, L is a sublattice of $C(\Omega)$ if for any $x, y \in L$, $x \vee y,\ x \wedge y \in L$. Any sublattice of $C(\Omega)$ has a positive basis. For more details on positive bases and lattice-subspaces we refer to [6] and [7].
Suppose that $\{b_1, b_2, ..., b_r\}$ is a positive basis of L. Then it is easy to show that for any $x = \sum_{i=1}^{r} \lambda_i b_i, y = \sum_{i=1}^{r} \mu_i b_i$ we have: $x \geq y$ if and only if $\lambda_i \geq \mu_i$ for each i. Also each b_i is an extremal point of L_+. (A vector $x_0 \in L_+, x_0 \neq 0$ is an extremal point of L_+ if for any $x \in L$, $0 \leq x \leq x_0$ implies $x = \lambda x_0$ for some real number λ). This property implies that a positive basis of L is unique in the sense of positive multiples. We give the next easy four results without proof. Recall that the support of a vector $x = (x(1), x(2), ..., x(m))$ of $\mathbb{R}^m$ is the set $supp(x) = \{i = 1, 2, ..., m | x(i) \neq 0\}$.

Proposition 13. *An ordered subspace Z of $C(\Omega)$ with a positive basis $\{b_1, b_2, ..., b_r\}$ is a sublattice of $C(\Omega)$ if and only if $b_i^{-1}(0, +\infty) \cap b_j^{-1}(0, +\infty) = \emptyset$ for any $i \neq j$.*

Proposition 14. *Suppose that Z is a finite dimensional sublattice of $C(\Omega)$. If the constant vector $\mathbf{1} = (1, 1, ..., 1)$ is an element of Z, then Z has a positive basis $\{b_1, b_2, ..., b_r\}$ which is a* **partition of the unit**, *i. e. $\mathbf{1} = \sum_{i=1}^{r} b_i$ and for each vector b_i we have: $b_i(t) = 1$, for each $t \in \Omega$ with $b_i(t) > 0$.*

Suppose now that $z_1, z_2, ..., z_r$ are fixed, linearly independent, positive vectors of $C(\Omega)$ and that Z is the subspace of $C(\Omega)$ generated by the vectors z_i. We present the results of [7] for the determination of a positive basis of the sublattice Z of $C(\Omega)$ generated by the set $\{z_1, z_2, ..., z_r\}$. The function

$$\beta(t) = \Big(\frac{z_1(t)}{z(t)}, \frac{z_2(t)}{z(t)}, ..., \frac{z_r(t)}{z(t)}\Big), \text{ for each } \ i \in \Omega, \text{ with } z(t) > 0,$$

where $z = z_1 + z_2 + ... + z_r$, is the **basic function** of $z_1, z_2, ..., z_r$. This definition which is given in [6], is very important for the study of positive bases. The set $R(\beta) = \{\beta(t) \big| t \in \Omega \ \text{ with } \ z(t) > 0\}$, is the range of β and the cardinal number $cardR(\beta)$ of $R(\beta)$ is the number of the (different) elements of $R(\beta)$. Under the above notations we have:

Theorem 15 ([7], Theorem 3.6). *Z is a sublattice of $C(\Omega)$ if and only if $cardR(\beta) = r$.*
If $R(\beta) = \{P_1, P_2, \dots, P_r\}$, a positive basis $\{b_1, b_2, ..., b_r\}$ of Z is given by the formula:

$$(b_1, b_2, ..., b_r)^T = A^{-1}(z_1, z_2, ..., z_r)^T, \tag{4}$$

where A is the $r \times r$ matrix whose the i^{th} column is the vector P_i, for each $i = 1, 2, ..., r$, and $(b_1, b_2, ..., b_r)^T$, $(z_1, z_2, ..., z_r)^T$ are the matrices with rows the vectors $b_1, b_2, ..., b_r$, $z_1, z_2, ..., z_r$.

The next result, [7], Theorem 3.7, gives an algorithm for the construction of the sublattice generated by a finite set of linearly independent and positive vectors. Statement (d) explains the way of construction of a positive basis by the previous theorem.

Theorem 16 ([7], Theorem 3.7). *Let Z be the sublattice of $C(\Omega)$ generated by $z_1, z_2, ..., z_r$ and let $m \in \mathbb{N}$. Then the statements (i) and (ii) are equivalent:*

(i) $dim(Z) = \mu$.

(ii) $R(\beta) = \{P_1, P_2, ..., P_\mu\}$.

If the statement (ii) is true then Z is constructed as follows:

(a) *Enumerate $R(\beta)$ so that its r first vectors to be linearly independent (as it is shown in [7], such an enumeration always exists). Denote again by $P_i,\ i = 1, 2, \dots, \mu$ the new enumeration and we put $I_{r+k} = \{t \in \Omega\} \big| \beta(t) = P_{r+k}\}$, for each $k = 1, 2, ..., \mu - r$.*

(b) *Define the vectors z_{r+k}, $k = 1, 2, ..., \mu - r$ as follows:*

$$z_{r+k}(i) = z(i) \ \text{ if } \ i \in I_{r+k} \ \text{ and } \ z_{r+k}(i) = 0 \ \text{ if } \ i \notin I_{r+k},$$

where $z = z_1 + z_2 + ... + z_r$ is the sum of the vectors z_i.

(c) $Z = [z_1, z_2, \dots, z_r, z_{r+1}, \dots, z_\mu]$.

(d) *A positive basis $\{b_1, b_2, ..., b_\mu\}$ of Z is constructed as follows:*

Consider the basic function γ of $z_1, z_2, \dots, z_r, z_{r+1}, \dots, z_\mu$ and suppose that $\{P'_1, P'_2, ..., P'_\mu\}$ is the range of γ (the range of γ has exactly μ points). Then

$$(b_1, b_2, ..., b_\mu)^T = D^{-1}(z_1, z_2, ..., z_\mu)^T, \tag{5}$$

where D is the $\mu \times \mu$ matrix with columns the vectors $P'_1, P'_2, ..., P'_\mu$.

References

[1] C. D. Aliprantis and R. Tourky, *Markets that don't replicate any option*, Economic Letters **76** (2002), 443–447.

[2] A. M. Baptista, *On the non-existence of redundant options*, Economic Theory **31** (2007), 205–212.

[3] C. Kountzakis and I. A. Polyrakis, *The completion of security markets*, Decisions in Economics and Finance **29** (2006), 1–21.

[4] S. F. LeRoy and J. Werner, *Principles of Financial Economics*, Cambridge University Press, 2001.

[5] C. Kountzakis I. A. Polyrakis and F. Xanthos, *Non replication of options*, to appear.

[6] I. A. Polyrakis, *Finite-dimensional lattice-subspaces of $C(\Omega)$ and curves of* $\mathbb{R}^n$, Transactions of the American Mathematical Society **384** (1996), 2793–2810.

[7] ———, *Minimal lattice-subspaces*, Transactions of the American Mathematical Society **351** (1999), 4183–4203.

[8] ———, *Linear Optimization in $C(\Omega)$ and Portfolio Insurance*, Optimization **52** (2003), 221–239.

[9] S. A. Ross, *Options and efficiency*, Quartely Journal of Economics **90** (1976), 75–89.