THE DUAL METHOD VIA THE ENVELOPE THEOREM FOR SENSITIVITY ANALYSIS AND LE CHATELIER PRINCIPLE IN PARAMETRIC OPTIMIZATION PROBLEMS

by

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Abstract

The paper develops the dual method, via the Envelope theorem, for sensitivity analysis and Le Chatelier Principle in general parametric optimization problem. A unified approach in matrix theoretic methods is exhibited that relies, not only on envelope tangencies, but mainly on curvature conditions of appropriate envelope problems suitable for each purpose.

If all parameters appear in all functions of the original problem then, no matter whether the associated envelope problem is constrained or not, curvature conditions are subject to constraints and cannot be used for either purpose; difficulties however are easily overcome by designing suitable representations of them that hold in the respective tangent subspaces. With a less general but more conducive parameter structure, unconstrained envelope problems lead directly to desired results.

The dual method is as elegant and simple as the primal method of sensitivity analysis and Le Chatelier Principle in matrix terms. All basic results are obtained equally well; the same difficulties appear and are similarly dealt with.

Finally, an economic application in the theory of consumer choice, without and with quantity rationing constraints, shows that the proposed dual method derives all known sensitivity results without any outside help. More importantly, it derives a richer vista of Le Chatelier effects, a feature due to more specific relationships between some parameters and some choice variables.

Keywords: Envelope tangencies and curvature conditions, Dual and Primal method for Sensitivity analysis and Le Chatelier principle.

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1. Introduction

In a recent paper – **Drandakis** (2003) – we considered **Caratheodory´s** (1935) theorem on the properties of the Inverse of the Bordered Hessian matrix of a general constrained optimization problem. After a proof of the theorem, in matrix theoretic terms, we proved a second theorem that compares the submatrices on the main diagonal of the Inverse matrix, before and after new and "just binding" constraints are introduced to the problem. Both theorems are instrumental in the **Primal method of comparative statics** for examining the sensitivity of the optimal solution and **Le Chatelier Principle**, as parameters of the problem vary.

The present paper focuses on the **Dual method** of sensitivity analysis, **via the Envelope Theorem**, in which parameters of the original problem become the choice variables while the former choice variables are treated as parameters. The first Envelope problem compares the value of the objective function, at the fixed optimal solution, to that of the optimal value function of the original problem, while the second compares the optimal value functions of both problems after and before the introduction of new constraints. The second-order conditions of both Envelope problems furnish the curvature conditions needed for sensitivity analysis and Le Chaterlier Principle.

Our aim is to produce a simple and unified formulation of the dual method that was originally presented in the two basic papers of **Silberberg** and **Hatta**. **Silberberg** (1974) examined a general constrained optimization problem, in which all parameters appear in both the objective and constraint functions. Although only the first Envelope problem is a constrained one, both problems lead to second-order conditions specifying that a certain matrix of the rates of change of the optimal solution(s) must be semi-definite, or definite **on the tangent subspace**. To use these restrictions for our purposes, we must specify **a matrix representation of the second-order conditions in the tangent subspace**, evaluate the matrix using the second-order derivative properties of the optimal value functions and simplify it as much as possible, so that the meaning of its semi-definiteness or definiteness becomes transparent. **Hatta** (1980), on the other hand, examines a similar but less general optimization problem in which, except for parameters appearing in the objective and constraint functions, constraint levels are not constant but may also vary. Here the first Envelope problem may be either a constrained one, needing a matrix representation in the tangent subspace, or an unconstrained problem leading directly to a semi-definite or definite matrix of the rates of change of the optimal solution. When new and just binding constraints are introduced, the second Envelope problem produces all desired results.

The paper is organized as follows. Section 2 considers the **Silberberg** and **Hatta** models and the dual method via the Envelope Theorem, while Section 3 presents Le Chatelier Principle for both models. Section 4 compares the basic features of the dual and primal methods, while Section 5 presents a simple but important economic application in the theory of consumer choice. There, the problems referred to in the previous paragraph have led to the use of an indirect dual method, which borrows relevant properties from a related optimization problem, the expenditure minimization problem. Our dual method needs no outside help and shows that both utility maximization and expenditure minimization can be used for proving sensitivity results in the other problem. Finally concluding remarks and references are given in Section 6.

Our analysis relies on classical optimization techniques in matrix theory terms. All vectors are treated as column vectors, unless they are enclosed within parentheses or appear as function arguments, while matrices are denoted by capital letters. Thus e.g. 0, 0_m, or $O_{m\kappa}$, indicate the zero scolar, a vector of m zeros, or an $m \times \kappa$ matrix of zeros, respectively, while $x_i(\alpha) \in R$ and $x(\alpha) \in R^n$ need no explanation, but $X_\alpha(\alpha)$ denotes the $n \times \kappa$ matrix of the partial derivatives of $x_i(\alpha)$ in α_j, $i = 1, \ldots, n$ and $j = 1, \ldots, \kappa$. Finally, a prime after a vector or a matrix denotes transposition.

2. The dual method of comparative statics via the Envelope Theorem

(a) The Silberberg model

Silberberg examines the problem

$$\phi(\alpha) \equiv \max_x \left\{ f(x,\alpha) \middle| h(x,\alpha)' = 0_m' \right\} , \tag{S}$$

where $x \in X \subset R^n$, $\alpha \in A \subset R^\kappa$ and both X, A are open sets. The real-valued functions $f, h^i \in C^2$, $i = 1,\ldots,m$, with f_x (x, α) , f_α (x, α) and ${h^i}_x(x,\alpha)$, ${h^i}_\alpha(x,\alpha)$ their gradient vectors and with H_x (x, α), H_α (x, α) denoting the m x n and m x κ gradient matrices of h (x, α) in x and α, respectively. Finally the dimensionality restrictions $1 \underset{=}{<} m < n \text{ or } \kappa$ are imposed to ensure the feasibility of (S) and of problem (E_S^c) below. A feasible solution of (S) is **regular** if

$$r\,(H_x\,(x,\alpha)) = m. \tag{R_x}$$

(S) is well behaved having interior solutions. If x^0 is regular and a local maximum of (S), there exist lagrangean multipliers $\lambda^{0\prime} \equiv (\lambda_1^0,\ldots,\lambda_m^0)$ and we have the necessary conditions:

$$f-o-c \left\{ f_x(x^0,\alpha) = \sum_{i=1}^{m} \lambda_i^0\, h_x^i(x^0,\alpha) \equiv H_x(x^0,\alpha)'\lambda^0 \ ,\ h(x^0,\alpha) = 0_n \right\} \tag{1}$$

and

$$s-o-n-c \left\{ \begin{matrix} \text{The hessian of (S) at } x^0,\lambda^0 \text{ is negative} \\ \text{semidefinite on the tangent subspace} \end{matrix} \right\} . \tag{2}$$

The hessian matrix is given by F_{xx} (x, α) – H_{xx} (x, λ. α), where $H_{xx}(x,\lambda,\alpha) \equiv \sum_{i=1}^{m} \lambda_i\, H_{xx}^i\,(x,\alpha)$ is the sum of the matrices of second derivatives of all h^i (x, α) in x , weighted by their respective lagrangean multipliers[1]. The tangent subspace at a regular x^0 is given by $T = \left\{ \eta \in R^n \ \middle|\ H_x(x^0,\alpha)\eta = 0_m \right\}$. On the other hand, if x^0, λ^0 satisfy (1) and if the

$$s-o-s-c \left\{ \begin{matrix} F_{xx}(x^0,\alpha) - H_{xx}(x^0,\lambda^0,\alpha) \text{ is negative definite} \\ \text{on } \{\eta \in R^n \ |H_x(x^0,\alpha)\ \eta = 0_m \ ,\ \eta \neq 0_n\} \end{matrix} \right\} \tag{2$'$}$$

are also satisfied, then x^0 attains a strict local maximum of (S). Finally, the Jacobian of (1) in x and –λ,

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[1] The difference between the n x n H_{xx} (x, λ, α) and the m x n gradient matrix H_x (x, α) should be noted.

$$\begin{bmatrix} F_{xx}(x,\alpha)-H_{xx}(x,\lambda,\alpha), & H_x(x,\alpha)' \\ H_x(x,\alpha), & O_{mm} \end{bmatrix} \equiv \begin{bmatrix} A\ , & B \\ B', & O_{mm} \end{bmatrix} \tag{3}$$

is the bordered Hessian of (S), which is an invertible matrix at a regular x^0 that satisfies (1) and (2′)[2]. Then the Implicit function theorem works and the solution vectors $x(\alpha)$, $\lambda(\alpha)$ are differentiable functions of $\alpha \in A$[3] .

(b) Derivative properties of the maximal value function

We can easily see that

$$\varphi_\alpha(\alpha) = f_\alpha(x(\alpha), \alpha) - H_\alpha(x(\alpha), \alpha)'\ \lambda(\alpha)\ , \tag{4}$$

if we use (1) and differentiate the constraints. We will also need the second-order derivative properties, which involve the symmetric matrix

$$\begin{aligned}\Phi_{\alpha\alpha}(\alpha) = {} & [F_{\alpha x}(x(\alpha),\alpha)-H_{\alpha x}(x(\alpha),\lambda(\alpha),\alpha)]X_\alpha(\alpha)+ \\ & +[F_{\alpha\alpha}(x(\alpha),\alpha)-H_{\alpha\alpha}(x(\alpha),\lambda(\alpha),\alpha)]- \\ & -H_\alpha(x(\alpha),\alpha)'\Lambda_\alpha(\alpha)\end{aligned} \tag{5}$$

and are rarely mentioned in the literature. Here again

$H_{\alpha x}(x(\alpha),\lambda(\alpha),\alpha) = \sum_{i=1}^{n} \lambda_i(\alpha) H^i_{\alpha x}(x(\alpha),\alpha)$ and similarly for $H_{\alpha\alpha}(x(\alpha),\lambda(\alpha),\alpha)$.

(c) First Envelope problem for (S)

With parameter values at α^0 , let $x^0 \equiv x(\alpha^0)$, $\lambda^0 \equiv \lambda(\alpha^0)$. Can we compare the difference, $f(x^0, \alpha) - \varphi(\alpha)$, of the value of the objective of (S) at x^0 from the maximal value function, as parameters vary around α^0 ? The answer is no, despite the facts that $f(x^0, \alpha^0) = \varphi(\alpha^0)$ and that $\varphi(\alpha)$ is the maximal value function of (S). The reason is that, $f(x^0, \alpha)$ as a function of $\alpha \in A$, does not necessarily respect the constraints $h(x^0, \alpha) = 0_m$. We must therefore consider **the constrained Envelope problem**

$$\max_{\alpha} \left\{ f(x^0,\alpha)-\phi(\alpha)\ \middle|\ h(x^0,\alpha)'=0'_m \right\}, \tag{E^c_S}$$

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[2] For a proof see e.g. **Drandakis** (2003), lemma 1.

[3] See e.g. **Luenberger** (1973), ch. 10.3, **Sundaram** (1996), as well as **Luenberger** (1969), chs 1-9.

in which the choice variables of (S) are the parameters and the parameters of (S) become the choice variables. It is obvious that (E^c_S) will be a well defined problem only if $\kappa > m$ and

$$r\,(H_\alpha\,(x\,(\alpha),\,\alpha)) = m \qquad (R_\alpha)$$

holds. Its solution is characterized by

$$f-o-c\,\left\{f_\alpha(x^0,\alpha) - \phi_\alpha(\alpha) - H_\alpha(x^0,\alpha)'\xi = 0_\kappa\ ,\ h(x^0,\alpha) = 0_m\right\}\ , \qquad (6)$$

which are satisfied at least at α^0 with $\xi^0 = \lambda^0$ – as we know from (4) – and by s-o-c involving the symmetric κ x κ matrix[4] $C^0 \equiv F^0_{\alpha\alpha} - \Phi^0_{\alpha\alpha} - H^0_{\alpha\alpha}$. But as we know from (5) C^0 is given by

$$C^0 \equiv -[F^0_{\alpha x} - H^0_{\alpha x}]X^0_\alpha + H^{0\prime}_\alpha \Lambda^0_\alpha$$

and so we have

$$s-o-c\left\{\begin{array}{c} -[F^0_{\alpha x} - H^0_{\alpha x}]X^0_\alpha \ \text{ is negative semi-definite} \\ \text{(or definite) on the tangent subspace} \\ \{\zeta \in R^\kappa \ |H^0_\alpha\ \zeta = 0_m \ \text{(and } \zeta \neq 0_\kappa)\ \} \end{array}\right\} \qquad (7)$$

The f-o-c in (6) are the **Envelope tangencies** at α^0, while the s-o-c in (7) ate the **Envelope curvature conditions** at α^0. It is clear that (E^c_S) attains a local maximum of zero at α^0, which may be strict if s-o-s-c are satisfied [5].

(6) and (7) correspond, respectively, to equations (6) and (10) in **Silberberg** (1974), where it is also noted that (10) are subject to constraints.

It is evident, however, that (7) cannot be used directly for comparative static analysis: indeed we know nothing about the κ x κ matrix, C^0 except that in the tangent subspace it must be negative semi-definite or definite. We cannot therefore escape the task of finding some $r \equiv \kappa - m$, $r > 0$, linearly independent vectors $\zeta', \ldots, \zeta^r$, forming a κ x r matrix, Z^0 , which provides a basis for all $\zeta \in R^\kappa$ such that $H^0_\alpha \zeta = 0_m$ and thus getting **a matrix representation** of (7), **in the tangent**

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[4] For simplicity, we omit function arguments in the following matrices. The superscript, o, indicates that (x^0, α) is evaluated at α^0.

[5] Since α^0 is known before solving (E^c_S), no problem is created if only s-o-n-c hold.

subspace, the $r \times r$ matrix $Z^{0\prime}C^0 Z^0$, that is simple enough and shows with sufficient transparency the meaning of having a negative semidefinite or definite product matrix.

In fact it is clear that with $r = \kappa - m > 0$, with (R_α) holding and with x^0 given, some m of the α_i 's can be uniquely determined from the rest α_i 's and x^0. Without any loss of generality, let us assume that our former vector of κ parameters, α, is now written as (α, γ), with $f(x^0, \alpha, \gamma)$ and $h(x^0, \alpha, \gamma) = 0_m$ and with $H_\alpha(x^0, \alpha^0)$ now given by $[H_\alpha(x^0, \alpha^0, \gamma^0), H_\gamma(x^0, \alpha^0, \gamma^0)]$. Then the constraints in (7) are now given by $\{(\eta, \theta)' \in R^\kappa \;|\, 0_m = H^0_\alpha \eta + H^0_\gamma \theta\}$ with H^0_γ an invertible $m \times m$ matrix. Then $0_m = H^{0-1}_\gamma H^0_\alpha \eta + \theta$, or $\theta = -H^{0-1}_\gamma H^0_\alpha \eta$ and we can form the $\kappa \times r$ matrix $Z^0 = \begin{bmatrix} I_{r\,x\,r} \\ -H^{0-1}_\gamma H^0_\alpha \end{bmatrix}$, with $r(Z^0) = r$ which satisfies the m constraints in (7) since

$$[H^0_\alpha, H^0_\gamma]Z^0 = [H^0_\alpha - H^0_\alpha] = O_{mr}.$$

It then remains the final step of deriving the product matrix and presenting it in the simplest possible way. Using (5) we show in Appendix A that

$$Z^{0\prime} C^0 Z^0 = \{-[F^0_{\alpha x} - H^0_{\alpha x}] + H^{0\prime}_\alpha H^{0-1\prime}_\gamma [F^0_{\gamma x} - H^0_{\gamma x}]\} \cdot [X^0_\alpha - X^0_\gamma H^{0-1}_\gamma H^0_\alpha], \tag{8}$$

with $\begin{bmatrix} F^0_{\alpha x} \\ F^0_{\gamma x} \end{bmatrix}, \begin{bmatrix} H^0_{\alpha x} \\ H^0_{\gamma x} \end{bmatrix}, \begin{bmatrix} X^0_\alpha, X^0_\gamma \end{bmatrix}$, or $[\Lambda^0_\alpha, \Lambda^0_\gamma]$ replacing our old $F^0_{\alpha x}$, $H^0_{\alpha x}$, X^0_α or Λ^0_α respectively. Thus we have from (E^c_S) **the curvature conditions for (S)**

$$s - o - c \left\{ \begin{array}{l} \text{The } r \times r \text{ matrix appearing in (8)} \\ \text{is negative semi-definite (or definite} \end{array} \right\}. \tag{8$'$}$$

Indeed for any $\eta \in R^r$, $Z^0 \eta = \begin{pmatrix} \eta \\ -H_\gamma^{0-1} \; H_\alpha^0 \, \eta \end{pmatrix} = \begin{pmatrix} \eta \\ \theta \end{pmatrix} \in R^\kappa$ and, since $[H_\alpha^0 \; ,H_\gamma^0] \, Z^0 \, \eta = 0_m$, we must have $\eta' Z^{0'} C^0 Z^0 \eta = (\eta \; ,\theta)' \, C^0 \begin{pmatrix} \eta \\ \theta \end{pmatrix} \leqq 0$, or $\eta' Z^0 C^0 Z^0 \eta < 0$ if s-o-s-c hold [6].

As for the second matrix appearing in (8), $[X_\alpha (\alpha^0, \gamma^0) - X_\gamma (\alpha^0, \gamma^0) \, H_\gamma (x^0, \alpha^0, \gamma^0)^{-1} H_\alpha (x^0, \alpha^0, \gamma^0)]$ has a clear meaning for us economists: it is the matrix of compensated variations in parameters α, as it shows the rates of change of $x (\alpha, \gamma)$ in α, when the remaining parameters, γ, also vary appropriately so as to keep the constraints $h (x^0, \alpha, \gamma) = 0_m$ satisfied at any such (α, γ). We can also see that the nullity of this matrix is at least equal to the number of constraints. Indeed differentiating $h (x (\alpha, \gamma) , \alpha, \gamma) \equiv 0_m$ w/r to α and γ we get $H_x X_\alpha + H_\alpha = O_{mr}$ and $H_x X_\gamma + H_\gamma = O_{mm}$. But from the latter we get $H_x X_\gamma \; H_\gamma^{-1} H_\alpha + H_\alpha = O_{mr}$, which when subtracted from the first produces

$$H_x [X_\alpha - X_\gamma \, H_\gamma^{-1} H_\alpha] = O_{m\,x\,r} \tag{9}$$

(d) The Hatta model

Hatta (1980) consider the optimization problem

$$\phi(\alpha,\gamma) \equiv \max_x \{ f(x,\alpha) \;\; | h(x,\alpha)' = \gamma' \} \quad , \tag{H}$$

with n choice variables, m constraints and $\kappa = r + m$ parameters (α, γ) in A, $r > 0$. Again $1 \leqq m < n$ and (R_x) are imposed. (H) is less general than (S), since parameters γ do not appear in the objective function and $h (x, \alpha) - \gamma = 0_m$ show that constraints here have a more specific structure. But as we shall see below, this is more than compensated by the additional results obtained.

The solution of (H) for $x (\alpha, \gamma)$ and $\lambda (\alpha, \gamma)$ satisfies

$$f-o-c \qquad \{ f_x(x,\alpha) = H_x(x,\alpha)'\lambda \;\; , \; h(x,\alpha) = \gamma \} \tag{10}$$

and

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[6] For the concept of a matrix that represents the s-o-c of an optimization problem in the tangent subspace see **Luenberger** (1973), chapter 10.4 .

$$\text{s}-\text{o}-\text{s}-\text{c}\left\{\begin{array}{l}\text{the hessian matrix } F_{xx}(x,\alpha)-H_{xx}(x,\lambda,\alpha)\\ \text{is negative definite on the subspace}\\ \{\eta\in R^{n}\ \left|H_{x}(x,\alpha)\eta=0_{m}\ ,\ \eta\neq 0_{n}\right\}\end{array}\right\} \tag{11}$$

and attains a strict local maximum of (H). Again the bordered Hessian of (H) is invertible and x (α, γ) and λ (α, γ) are differentiable functions in A.

(e) **Derivative properties of φ (α, γ)**

The maximal value function $\varphi(\alpha,\gamma)\equiv f(x(\alpha,\gamma),\alpha)$ has first-order derivatives

$$\varphi_\alpha(\alpha,\gamma)=f_\alpha(x(\alpha,\gamma),\alpha)-H_\alpha(x(\alpha,\gamma),\alpha)'\,\lambda(\alpha,\gamma)\ , \tag{12}$$

$$\varphi_\gamma(\alpha,\gamma)=\lambda(\alpha,\gamma) \tag{13}$$

and thus

$$\varphi_\alpha(x(\alpha,\gamma),\alpha)+H_\alpha(x(\alpha,\gamma),\alpha)'\,\varphi_\gamma(\alpha,\gamma)=f_\alpha(x(\alpha,\gamma),\alpha)\quad , \tag{14}$$

as well as, second-order derivative properties involving the symmetric matrix

$$\begin{bmatrix}\Phi_{\alpha\alpha}(\alpha,\gamma), & \Phi_{\alpha\gamma}(\alpha,\gamma)\\ \Phi_{\gamma\alpha}(\alpha,\gamma), & \Phi_{\gamma\gamma}(\alpha,\gamma)\end{bmatrix}=\begin{bmatrix}[F_{\alpha x}-H_{\alpha x}]X_\alpha+[F_{\alpha\alpha}-H_{\alpha\alpha}]-H'_\alpha\Lambda_\alpha & , & [F_{\alpha x}-H_{\alpha x}]X_\gamma-H'_\alpha\Lambda_\gamma\\ \Lambda_\alpha & , & \Lambda_\gamma\end{bmatrix} \tag{15}$$

(f) First Envelope problem for (H)

For any $(\alpha^0,\gamma^0)\in A$, let $x^0=x(\alpha^0,\gamma^0)$ and $\lambda^0=\lambda(\alpha^0,\gamma^0)$ and consider **the constrained Envelope problem**

$$\max_{\alpha,\gamma}\{f(x^0,\alpha)-\phi(\alpha,\gamma)\ \left|h(x^0,\alpha)'=\gamma'\right.\} \tag{E^c_H}$$

with $1\leqq m<\kappa=r+m$ and (R_α) holding. Quite briefly we have :

$$f-o-c\left\{f_\alpha(x^o,\alpha)-\varphi_\alpha(\alpha,\gamma)-H_\alpha(x^o,\alpha)'\xi=0_r\,,-\varphi_\gamma(\alpha,\gamma)+\xi=0_m\,,h(x^o,\alpha)=\gamma\right\} \tag{16}$$

which are satisfied at least at (α^0,γ^0) with $\xi^0=\lambda^0$, while the Hessian of (E^c_H) at (α^0,γ^0), or

$$\begin{bmatrix}F^0_{\alpha\alpha}-\Phi^0_{\alpha\alpha}-H^0_{\alpha\alpha}, & -\Phi^0_{\alpha\gamma}\\ -\Phi^0_{\gamma\alpha}, & -\Phi^0_{\gamma\gamma}\end{bmatrix}=\begin{bmatrix}-[F^0_{\alpha x}-H^0_{\alpha x}]X^0_\alpha+H^{0\prime}_\alpha\Lambda^0_\alpha & , & -[F^0_{\alpha x}-H^0_{\alpha x}]X^0_\alpha+H^{0\prime}_\alpha\Lambda^0_\gamma\\ -\Lambda^0_\alpha & , & -\Lambda^0_\gamma\end{bmatrix} \tag{17a}$$

must be n- s- d (or n – d) on the tangent subspace

$$\left\{(\eta,\theta)'\in R^{r+m}\ \middle|\ [H^0_\alpha,-I_{mm}](\eta,\theta)'=0_m\ ,\ ((\eta,\theta)\neq 0'_{r+m})\right\}. \tag{17b}$$

Multiplying both sides of (17a) by the (r+m) x r matrix $Z^0=\begin{bmatrix} I_{rr} \\ H^0_\alpha \end{bmatrix}$ and its transpose, we get **the matrix representation of the s-o-c** of (E^c_H) **in the tangent subspace**, $-[F^0_{\alpha x}-H^0_{\alpha x}][X^0_\alpha+X^0_\gamma H^0_\alpha]$ and **the curvature conditions**

$$s-o-c\left\{\begin{array}{c} -[F^0_{\alpha x}-H^0_{\alpha x}][X^0_\alpha+X^0_\gamma H^0_\alpha]\ \text{is} \\ \text{a negative semi}-\text{definite} \\ \text{(or definite) matrix} \end{array}\right\}\ . \tag{18}$$

It is possible, however, to explore the specific structure of h (x, α) = γ and consider **the unconstrained Envelope problem**

$$\max_\alpha\{f(x^0,\alpha)-\varphi(\alpha,h(x^0,\alpha))\} \tag{E_H}$$

with

$$f-o-c\{f_\alpha(x^0,\alpha)-\varphi_\alpha(\alpha,h(x^0,\alpha))-H_\alpha(x^0,\alpha)'\varphi_\gamma(\alpha,h(x^0,\alpha))=0_r\}\ , \tag{19}$$

which are satisfied at least at α^0 and $h(x^0,\alpha^0)=\gamma^0$,
as well as with

$$s-o-c\left\{\begin{array}{c} \text{The matrix } F^0_{\alpha\alpha}-\Phi^0_{\alpha\alpha}-\Phi^0_{\alpha\gamma}H^0_\alpha- \\ -H^{0\prime}_\alpha\Phi^0_{\gamma\alpha}-H^{0\prime}_\alpha\Phi^0_{\gamma\gamma}H^0_\alpha-H_{\alpha\alpha}(x^0,\varphi^0_\gamma,\alpha^0) \\ \text{is negative semi}-\text{definite (or definite)} \end{array}\right\}\ . \tag{20}$$

At α^0, we attain a (strict) local maximum of zero (if s-o-s-c hold). Using than the second-order derivatives in (15), we can easily show that the s-o-c of (E_h) are given by

$$s-o-c\left\{\begin{array}{c} \text{The r x r matrix} \\ -[F^0_{\alpha x}-H^0_{\alpha x}][X^0_\alpha+X^0_\gamma H^0_\alpha]\ \text{is negative} \\ \text{semi}-\text{definite (or definite)} \end{array}\right\} \tag{20′}$$

in complete conformity with (18) for (E^C_H)[7] .

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[7] An alternative route to (19) – (20′) is also possible if, like **Hatta** we consider the compensated version of (H) itself, (H_{comp}), and proceed from there. This too is quite interesting and is given in Appendix B.

Before closing this section, let us point out that the matrix in (8), which is the representation of the s-o-c of (E_S^c), is reduced to that in either (18) or (20′) when $h(x, \alpha, \gamma) = 0_m$ becomes $h(x, \alpha) - \gamma = 0_m$. Then $H_\gamma(x, \alpha, \gamma) = - I_{mm}$ and thus matrix $[X_\alpha^0 - X_\gamma^0 \, H_\gamma^{0-1} \, H_\alpha^0]$ reduces to $[X_\alpha^0 + X_\gamma^0 \, H_\alpha^0]$.

Finally, we note that $H_x(x(\alpha, \gamma), \alpha)\,[X_\alpha(\alpha, \gamma) + X_\gamma(\alpha, \gamma)\, H_\alpha(x(\alpha, \gamma), \alpha)] = O_{m \times r}$, by differentiating $h(x(\alpha,\gamma),\alpha) = \gamma \quad w/r \ \text{to} \ (\alpha,\gamma)$. Then we also get

$$\begin{aligned} & f_x(x(\alpha,\gamma),\alpha)'[X_\alpha(\alpha,\gamma) + X_\gamma(\alpha,\gamma)\, H_\alpha(x(\alpha,\gamma),\alpha)] = \\ & = \lambda(\alpha,\gamma)' H_x(x(\alpha,\gamma),\alpha)[X_\alpha(\alpha,\gamma) + X_\gamma(\alpha,\gamma) H_\alpha(x(\alpha,\gamma),\alpha)] = 0'_r . \end{aligned} \tag{21}$$

3. Optimization Problem with additional Constraints: Le Chatelier Principle

Let us suppose that a new problem, similar to (S) but with additional constraints, $h^+(x,\alpha) = 0_{m^+}$ is to be considered in relation to (S). This “second-best” problem is given by

$$\begin{aligned} \tilde{\phi}(\alpha) & \equiv \max_x \left\{ f(x,\alpha) \middle| h(x,\alpha)' = 0'_m , h^+(x,\alpha)' = 0'_{m^+} \right\} \\ & \equiv \max_x \left\{ f(x,\alpha) \middle| \tilde{h}(x,\alpha)' = 0'_{m'} \right\}, \end{aligned} \tag{$\tilde{S}$}$$

with $m + m^+ = m' < n$ or κ, $\tilde{h}(x,\alpha)' = (h(x,\alpha) \, , \, h^+(x,\alpha))$ and

$$r(\tilde{H}_x(x,\alpha)) = m' \tag{$\tilde{R}_x$}$$

imposed.

The solution of $(\tilde{S})$, $\tilde{x}(\alpha)$ and $\tilde{\lambda}(\alpha)' = (\hat{\lambda}(\alpha) \, , \, \hat{\lambda}^+(\alpha))$, is derived from (1) and (2′) appropriately modified so as to incorporate the new constraints[8]. We also have $\tilde{\varphi}(\alpha) \equiv f(\tilde{x}(\alpha),\alpha)$ with

$$\tilde{\varphi}_\alpha(\alpha) = f_\alpha(\tilde{x}(\alpha),\alpha) - \tilde{H}_\alpha(\tilde{x}(\alpha),\alpha)' \, \tilde{\lambda}(\alpha) \tag{22}$$

and

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[8] Here $\tilde{H}_x(\tilde{x}(\alpha),\alpha)' = [\hat{H}_x(\tilde{x}(\alpha),\alpha)' \, , \hat{H}_x^+(\tilde{x}(\alpha),\alpha)'], \tilde{H}_{\alpha x}(\tilde{x}(\alpha),\tilde{\lambda}(\alpha),\alpha) \equiv \sum_{i=1}^{m} \hat{\lambda}_i(\alpha) \hat{H}_{\alpha x}^i(\tilde{x}(\alpha),\alpha) + + \sum_{i=1}^{m+} \hat{\lambda}_i^+(\alpha) \, \hat{H}_{\alpha x}^{+i}(\tilde{x}(\alpha),\alpha)$, $\tilde{F}_{\alpha x}(\tilde{x}(\alpha),\alpha)$, etc.

$$\tilde{\Phi}_{\alpha\alpha}(\alpha) = [\tilde{F}_{\alpha\alpha} - \tilde{H}_{\alpha\alpha}]\tilde{X}_{\alpha} + [\tilde{F}_{\alpha\alpha} - \tilde{H}_{\alpha\alpha}] - \tilde{H}'_{\alpha}\tilde{\Lambda}_{\alpha} \tag{23}$$

In general (S) and $(\tilde{S})$ have different solutions, with $\phi(\alpha) > \tilde{\phi}(\alpha)$ and equality appearing when the additional constraints are just binding at some parameter values, α^0[9]. Then $x(\alpha^0) = \tilde{x}(\alpha^0)$ and from the f-o-c of both problems we see that

$$\begin{aligned} &f_x(x(\alpha^0),\alpha^0) - H_x(x(\alpha^0),\alpha^0)'\lambda(\alpha^0) = f_x(x(\alpha^0),\alpha^0) - \\ &-H_x(x(\alpha^0),\alpha^0)'\ \hat{\lambda}(\alpha^0) - H_x^+(x(\alpha^0),\alpha^0)'\ \hat{\lambda}^+(\alpha^0) \end{aligned}$$

reduces to

$$H_x(x(\alpha^0),\alpha^0)'\{\lambda(\alpha^0) - \hat{\lambda}(\alpha^0)\} = H_x^+(x(\alpha^0),\alpha^0)'\hat{\lambda}^+(\alpha^0)\,.$$

But this implies that $\lambda(\alpha^0) = \hat{\lambda}(\alpha^0)$ and $\hat{\lambda}^+(\alpha^0) = 0_{m^+}$ because of $(\tilde{R}_x)$.

We can then consider the **second Envelope problem for (S) and $(\tilde{S})$**, i.e.,

$$\max_{\alpha}\{\tilde{\varphi}(\alpha) - \varphi(\alpha)\}\ , \tag{$E_{S\tilde{S}}$}$$

which is unconstrained since the impact of $h^+(x, \alpha) = 0_m{}^+$ is incorporated into $\tilde{\varphi}(\alpha)$. $(E_{S\tilde{S}})$ is characterized by

$$f-o-c\ \{\tilde{\varphi}_{\alpha}(\alpha) - \varphi_{\alpha}(\alpha) = 0_{\kappa}\} \tag{24}$$

and

$$s-o-c\left\{\begin{matrix}\tilde{\Phi}_{\alpha\alpha}(\alpha) - \Phi_{\alpha\alpha}(\alpha)\ \text{ is a negative} \\ \text{semi}-\text{definite (or definite) matrix}\end{matrix}\right\}\ , \tag{25}$$

satisfied at least at α^0 and attaining a local maximum of zero there, which may be strict when s-o-s-c hold.

But the matrix in (25), evaluated at α^0, becomes equal to[10]

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[9] If inequality constraints were present in (S) and $(\tilde{S})$, then $\phi(\alpha^0) = \tilde{\phi}(\alpha^0)$ could also occur if the additional constraints were not binding at α^0.

[10] **Silberberg** (1974) considers Le Chatelier Principle for finite variations of α around α^0. Using a Taylor's series expansion he ends up with (19), which correspond to our (25) – (25′). Note also that $f(x^0,\alpha) - \varphi(\alpha) - \{f(x^0,\alpha) - \tilde{\varphi}(\alpha)\} = \tilde{\varphi}(\alpha) - \varphi(\alpha)$.

$$\begin{aligned}&\tilde{\Phi}_{\alpha\alpha}(\alpha^0)-\Phi_{\alpha\alpha}(\alpha^0)=[F^0_{\alpha x}-H^0_{\alpha x}][\tilde{X}^0_{\alpha}-X^0_{\alpha}]-\\ &-\tilde{H}^{0\prime}_{\alpha}[\tilde{\Lambda}^0_{\alpha}-\begin{bmatrix}\Lambda^0_{\alpha}\\ O_{m^+m^+}\end{bmatrix}]\end{aligned}\tag{25′}$$

and leads to

$$s-o-c\left\{\begin{array}{l}[F^0_{\alpha x}-H^0_{\alpha x}][\tilde{X}^0_{\alpha}-X^0_{\alpha}]\text{ is negative}\\ \text{semi-definite (or definite) on the}\\ \text{subspace }\{\zeta\in R^{\kappa}\ |\tilde{H}^0_{\alpha}\ \zeta=0_{m'}\ (\zeta\neq 0_{\kappa})\}\end{array}\right\}\tag{25′′}$$

It is evident again that we must derive a matrix representation of the κ x κ $[F^0_{\alpha x}-H^0_{\alpha x}][\tilde{X}^0_{\alpha}-X^0_{\alpha}]$ matrix in its tangent subspace. Following the same method that we used in § 2 (c), we write our κ parameters $\alpha'=(\alpha_1,\ldots,\alpha_\kappa)$ as $(\alpha,\tilde{\gamma})$ with $\tilde{\gamma}'=(\gamma_1,\ldots,\gamma_{m'})$, $\alpha'=(\alpha_1,\ \ldots,\ \alpha_{r'})$ and $r'+m'=\kappa$, $r'>0$, as well as with $f(x^0,\alpha,\tilde{\gamma})$, $\tilde{h}(x^0,\alpha,\tilde{\gamma})$ and with $\tilde{H}_{\alpha}(x^0,\alpha)$ written as $[\tilde{H}_{\alpha}(x^0,\alpha,\tilde{\gamma}),\tilde{H}_{\tilde{\gamma}}(x^0,\alpha,\tilde{\gamma})]$. Since the m' x m' matrix $\tilde{H}_{\tilde{\gamma}}(x^0,\alpha,\tilde{\gamma})$ is invertible, we can form the κ x r' $\tilde{Z}^0=\begin{bmatrix}I_{r'xr'}\\ -\tilde{H}^{0-1}_{\tilde{\gamma}}\tilde{H}^0_{\alpha}\end{bmatrix}$ matrix and multiply both sides of the matrix in (25) by Z^0 and its transpose. As shown in Appendix C we then get the r' x r' matrix

$$\begin{aligned}&\{F^0_{\alpha x}-H^0_{\alpha x}-\tilde{H}^{0\prime}_{\alpha}\tilde{H}^{0'^{-1}}_{\tilde{\gamma}}[F^0_{\tilde{\gamma}\alpha}-H^0_{\tilde{\gamma}x}]\}\{[\tilde{X}^0_{\alpha}-\tilde{X}^0_{\tilde{\gamma}}\tilde{H}^{0^{-1}}_{\tilde{\gamma}}\tilde{H}^0_{\alpha}]-\\ &-[X^0_{\alpha}-X^0_{\tilde{\gamma}}\tilde{H}^{0^{-1}}_{\tilde{\gamma}}\tilde{H}^0_{\alpha}]\}\end{aligned}\tag{26}$$

and **the curvature conditions for (S) and $(\tilde{\mathbf{S}})$** are given by

$$s-o-c\left\{\begin{array}{c}\text{The matrix in (26) is negative}\\ \text{semi-definite (or definite)}\end{array}\right\}\tag{26′}$$

Turning to (H) let us consider the “second-best” problem

$$\tilde{\varphi}(\alpha,\tilde{\gamma})\equiv\max_{x}\{f(x,\alpha)\ |\tilde{h}(x,\alpha)'=\tilde{\gamma}'\},\tag{$\tilde{H}$}$$

in which there are $m+m^+=m'<n$ constraints and $\kappa'=r+m'$ parameters, with $\kappa'-\kappa=m^+$, while

$$r(\tilde{H}_x(x,\alpha))=m'\tag{$\tilde{R}_x$}$$

also holds. The solution of $(\tilde{H})$, $\tilde{x}(\alpha,\tilde{\gamma})$ and $\tilde{\lambda}(\alpha,\tilde{\gamma})' = (\hat{\lambda}(\alpha,\tilde{\gamma}) \ , \hat{\lambda}^{+}(\alpha,\tilde{\gamma}))$, satisfies (10) and (11) above appropriately modified, while $\tilde{\varphi}(\alpha,\tilde{\gamma}) \equiv f(\tilde{x}(\alpha,\tilde{\gamma}),\alpha)$ has first and second-order derivative properties corresponding to those in (12) – (14) and (15) and, finally, the unconstrained Envelope problem

$$\max_{\alpha}\{f(x^{0},\alpha) - \tilde{\varphi}(\alpha,\tilde{h}(x^{0},\alpha))\} \qquad (E_{\tilde{H}})$$

leads to f-o-c and s-o-c corresponding to those in (19) and (20′) above.

Again $\varphi(\alpha,\gamma) > \tilde{\varphi}(\alpha,\tilde{\gamma})$, in general, but if we assume that $x(\alpha^{0},\gamma^{0}) = \tilde{x}(\alpha^{0},\tilde{\gamma}^{0})$ at $(\alpha^{0},\tilde{\gamma}^{0})$, then $\lambda(\alpha^{0},\gamma^{0}) = \hat{\lambda}(\alpha^{0},\tilde{\gamma}^{0})$, $\hat{\lambda}^{+}(\alpha^{0},\tilde{\gamma}^{0}) = 0_{m^{+}}$ and $\varphi(\alpha^{0},\gamma^{0}) = \tilde{\varphi}(\alpha^{0},\tilde{\gamma}^{0})$.

We can then proceed and consider **the second Envelope problem for (H) and $(\tilde{\mathbf{H}})$**

$$\max_{\alpha}\{\tilde{\varphi}(\alpha,\tilde{h}(x^{0},\alpha) - \varphi(\alpha,h(x^{0},\alpha)\} \ , \qquad (E_{H\tilde{H}})$$

which attains a local maximum of zero at $(\alpha^{0},\tilde{\gamma}^{0})$ with

$$\text{f}-\text{o}-\text{c} \qquad \{\tilde{\varphi}_{\alpha^{0}} + \tilde{H}_{\alpha}^{0\prime}\tilde{\varphi}_{\tilde{\gamma}}^{0} - \varphi_{\alpha}^{0} - H_{\alpha}^{0\prime}\varphi_{\gamma}^{0} = 0_{r}\} \qquad (27)$$

and

$$\text{s}-\text{o}-\text{c} \quad \left\{\begin{array}{c} \text{matrix } \{[\tilde{\Phi}_{\alpha\alpha}^{0} + \tilde{\Phi}_{\alpha\tilde{\gamma}}^{0}\tilde{H}_{\alpha}^{0}] + [\tilde{H}_{\alpha\alpha}^{0} + \tilde{H}_{\alpha}^{0\prime}\tilde{\Phi}_{\tilde{\gamma}\alpha}^{0} + \tilde{H}_{\alpha}^{0\prime}\tilde{\Phi}_{\tilde{\gamma}\tilde{\gamma}}^{0}\tilde{H}_{\alpha}^{0}] - \\ - \ [\Phi_{\alpha\alpha}^{0} + \Phi_{\alpha\gamma}^{\prime 0}H_{\alpha}^{0}] - [H_{\alpha\alpha}^{0} + H_{\alpha}^{0\prime}\Phi_{\gamma\alpha}^{0} + H_{\alpha}^{0}\Phi_{\gamma\gamma}^{0}H_{\alpha}^{0}] \\ \text{is negative semi}-\text{definite (or definite)} \end{array}\right\} \qquad (28)$$

At (α^{0}, γ^{0}, γ^{+0}), however, $\tilde{H}_{\alpha}^{0\prime}\tilde{\varphi}_{\tilde{\gamma}}^{0} = H_{\alpha}^{0\prime}\varphi_{\gamma}^{0}$ and so we get the far simpler conditions

$$\text{f}-\text{o}-\text{c} \ \{\tilde{\varphi}_{\alpha}(\alpha^{0},\tilde{h}(x^{0},\alpha^{0})) - \varphi_{\alpha}(\alpha^{0},h(x^{0},\alpha^{0})) = 0_{r}\} \qquad (27')$$

and **the curvature conditions for (H) and** $(\tilde{H})$

$$\text{s}-\text{o}-\text{c} \quad \left\{\begin{array}{c} \text{matrix } [F_{\alpha x}^{0} - H_{\alpha x}^{0}]\{\tilde{X}_{\alpha}^{0} + \tilde{X}_{\tilde{\gamma}}^{0}\tilde{H}_{\alpha}^{0}] - \\ -[X_{\alpha}^{0} + X_{\gamma}^{0}H_{\alpha}^{0}]\} \\ \text{is negative semi}-\text{definite(or definite)} \end{array}\right\} , \qquad (28')$$

after using the second-order derivative properties of $\tilde{\Phi}^0_{\alpha\alpha}$, $\tilde{\Phi}^0_{\alpha\tilde{\gamma}}$, $\Phi^0_{\alpha\alpha}$ and $\Phi^0_{\alpha\gamma}$ and cancelling out equal terms with opposite signs.

4. Basic Features of the Primal and Dual Methods for Sensitivity Analysis and Le Chatelier Principle

Having completed the dual method for sensitivity analysis and Le Chatelier Principle, it is advisable to compare its main features to those of the Primal method.

The primal method can be based on a theorem of **Caratheodory** (1935, ch. 11 on ordinary maxima and minima) on the properties of the Inverse of the Bordered Hessian matrix. Without computing the Inverse $\begin{bmatrix} U, & V \\ V', & W \end{bmatrix}$ of $\begin{bmatrix} A, & B \\ B', & O_{mn} \end{bmatrix}$, it can be easily shown that the n x n U matrix is negative semidefinite with r(U) = n – m, the m x m matrix W = -V′ AV is positive (semi)definite if A itself happens to be negative (semi)definite, while r((V) = m and V is a pseudoinverse of B′ [11]. These results are due to the maximization hypothesis, before we consider parameters explicitly and the sensitivity of optimal solution to their variations.

The Primal method starts by differentiating the f-o-c of a behavioral system like (S) or (H) in their respective parameters and solving for the unknown rates of change of the choice variables.

Thus in (S) we see from (1) that[12]

$$\begin{bmatrix} A, & B \\ B', & O_{mm} \end{bmatrix} \begin{bmatrix} X_\alpha(\alpha) \\ -\Lambda_\alpha(\alpha) \end{bmatrix} = \begin{bmatrix} -[F_{x\alpha}(x(\alpha),\alpha) - H_{x\alpha}(x(\alpha),\lambda(\alpha),\alpha)] \\ -H_\alpha(x(\alpha),\alpha) \\ \end{bmatrix}, \qquad (29)$$

or

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[11] See **Drandakis** (2003). When m = n, then U = O_{nn} , W = $-B^{-1}\ A\ B^{-1'}$ and V is the inverse of B′.

[12] (29) is called the fundamental matrix equation for comparative static analysis and was introduced by **Barten** in 1966. For an early application to consumer theory, see **Barten, Kloek** and **Lampars** (1969).

$$\begin{bmatrix} X_\alpha(\alpha) \\ -\Lambda_\alpha(\alpha) \end{bmatrix} = \begin{bmatrix} U, & V \\ V', & W \end{bmatrix} \begin{bmatrix} -[F_{x\alpha} - H_{x\alpha} \\ -H_\alpha \end{bmatrix} =$$

$$= \begin{bmatrix} -U[F_{x\alpha} - H_{x\alpha}] - VH_\alpha \\ V'[F_{x\alpha} - H_{x\alpha}] - WH_\alpha \\ \end{bmatrix} \tag{30}$$

and so we get **the curvature conditions**

$$\left\{ \begin{array}{c} \text{The } \kappa \times \kappa \text{ matrix} \\ -[F_{\alpha x} - H_{\alpha x}]X_\alpha(\alpha) = [F_{\alpha x} - H_{\alpha x}]U[F_{x\alpha} - H_{x\alpha}] - \\ -[F_{x\alpha} - H_{x\alpha}]VH_\alpha = [F_{\alpha x} - H_{\alpha x}]U[F_{x\alpha} - H_{x\alpha}] \\ \text{is negative senidefinite (or definite)} \\ \text{on } \{\zeta \in R^\kappa \mid H_\alpha\, \zeta = 0_m \quad (\text{and} \quad \zeta \neq 0_\kappa)\} \end{array} \right\} \tag{31}$$

Clearly (31), as well as (7) above, cannot be used directly for comparative static analysis since they specify curvature conditions which are subject to constraints. It is evident, therefore, that a matrix representation of the curvature conditions in the tangent subspace is needed, **irrespective of whether the dual or the primal method is used**. Again with f (x, α) and h (x, α) written as f (x, α, γ) and h (x, α, γ), as we did above in §2, it can be easily seen that (31) is transformed into **the curvature conditions for (S)**.

$$\left\{ \begin{array}{c} \text{The } r \times r \text{ matrix} \\ \{-[F_{\alpha x} - H_{\alpha x}] + H'_\alpha H_\gamma^{-1\prime}[F_{\gamma x} - H_{\gamma x}]\}[X_\alpha - X_\gamma H_\gamma^{-1} H_\alpha] = \\ = \{[F_{\alpha x} - H_{\alpha x} - H'_\alpha H_\gamma^{-1\prime}[F_{\gamma x} - H_{\gamma x}]\}\, U\, \{F_{x\alpha} - H_{x\alpha} - [F_{x\gamma} - H_{x\gamma}] \cdot H_\gamma^{-1} H_\alpha] \\ \text{is negative semidefinite (or definite)} \end{array} \right\} . \tag{32}$$

Similarly in (H) we see from (10) above that

$$\begin{bmatrix} X_\alpha, & X_\gamma \\ -\Lambda_\alpha, & -\Lambda_\gamma \end{bmatrix} = \begin{bmatrix} U, & V \\ V', & W \end{bmatrix} \begin{bmatrix} -[F_{x\alpha} - H_{x\alpha}], & O_m \\ -H_\alpha\,, & I_{mm} \end{bmatrix} =$$

$$= \begin{bmatrix} -U[F_{x\alpha} - H_{x\alpha}] - VH_\alpha\,, & V \\ -V'[F_{x\alpha} - H_{x\alpha}] - WH_\alpha\,, & W \end{bmatrix} \quad , \tag{33}$$

or, rearranging terms, that

$$\begin{bmatrix} X_\alpha + X_\gamma H_\alpha \\ -\Lambda_\alpha + \Lambda_\gamma H_\alpha \end{bmatrix} = \begin{bmatrix} -U[F_{x\alpha} - H_{x\alpha}] \\ -V'[F_{x\alpha} - H_{x\alpha}] \end{bmatrix} . \tag{34}$$

Thus **the curvature conditions for (H)** are given by

$$\left\{ \begin{array}{l} -[F_{\alpha x} - H_{\alpha x}][X_\alpha + X_\gamma H_\alpha] = \\ = [F_{\alpha x} - H_{\alpha x}]U[F_{x\alpha} - H_{x\alpha}] \\ \text{is negative semidefinite (or definite)} \end{array} \right\} \tag{35}$$

exactly as in (18) or (20′) above.

When, finally, new constraints are added to either (S) or (H), which happen to be just binding at some x, the comparable submatrices on the main diagonal of the Inverse of the Bordered Hessian, before and after the introduction of new constraints, have a structural property that is at the care of Le Chatelier Principle. As shown in **Drandakis** (2003) the same curvature conditions, as in (26) for (S) and (28′) for (H), are produced.

In conclusion, both methods of sensitivity analysis are quite elegant in matrix theoretic terms. The primal method appears to be easier and has the advantage that it stresses some structural properties of the optimization hypothesis that are independent of the number of parameters and of the way that they appear in the problem. On the other hand, however, the dual method, utilizes the s-o-c of constrained or unconstrained Envelope problems and determines the appropriate curvature conditions in the tangent subspace that are needed for sensitivity analysis and Le Chatelier Principle. In addition, the Envelope theorem has further important uses, some of which will be touched on in the following section.

5. An Economic application: The effects of quantity rationing in the Theory of Consumer Choice

The dual method via the Envelope theorem has been applied to the theory of consumer choice in **Drandakis** (2007). A typical consumer is assumed to maximize his utility function $f(x)$, $x \in X \subset R^{n}_{++}$, $f \in C^{z}$, subject to an income constraint $w'x = y$, where $w > 0_n$ is the price vector and $y > 0$ is income devoted to consumption. Under quantity rationing of some goods, f(x) is written as $f(x,z)$, $x \in X^{r} \subset R^{\ell}_{++}$, $z \in Z^{m} \subset R^{m}_{++}$ and $\ell + m = n$, while to $w'x + r'z = y$ are added m new constraints $z = \overline{z}$.

We have then for the "first-best" problem

$$v(w,y) \equiv \max_{x}\{f(x) \;|\, w'x = y\} \qquad (U^{f})$$

while the "second-best" optimization problem is given by

$$v(w, y - r'\overline{z}) \equiv \max_{x}\{f(x,\overline{z}) \;|\, w'x = y - r'\overline{z}\}\ ,$$

or (U^{s})

$$v(w,r,y,\overline{z}) \equiv \max_{x,z}\{f(x,z) \;|\, w'x + r'z = y \ ,\ z = \overline{z}\}\ .$$

The solution of (U^{f}) and (U^{s}) is $x\,(w,\ y) > 0_m$, $\lambda\,(w,\ y) > 0$ and $x(w, y - r'\overline{z}) > 0_{\ell}$, $\lambda(w, y - r'\overline{z}) > 0$, or $x(w,r,y,\overline{z}) > 0_{\ell}$, $z(w,r,y,\overline{z}) = \overline{z} > 0_m$, $\lambda(w,r,y,\overline{z}) > 0$, $\mu(w,r,y,\overline{z}) \overset{>}{\underset{<}{=}} 0_m$, respectively[13, 14].

The first Envelope problems in both (U^{f}) and (U^{s}) lead to curvature conditions

$$\left\{\begin{array}{c} \text{The } n \,x\, n \text{ matrix} \\ [X_w(w,y) + x_y(w,y)\,x(w,y)'] \\ \text{is negative semi-definite (or definite} \\ \text{for } \eta \neq 0_n \text{ and } \eta \neq tw,\ t > 0) \end{array}\right\}$$

and

$$\left\{\begin{array}{c} \text{The } \ell \,x\, \ell \text{ matrix} \\ [X_w(w,r,y,\overline{z}) + x_y(w,r,y,\overline{z})\,x(w,r,y,\overline{z})'] \\ \text{is negative semi-definite (or definite} \\ \text{for } \eta \neq 0_{\ell} \text{ and } \eta \neq tw,\ t > 0) \end{array}\right\}$$

respectively, no matter whether they are constrained problems and need a matrix representation in the tangent subspace or are unconstrained.

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[13] Due to the equality rationing constraints, their respective multipliers, $\mu_j(w,r,y,\overline{z})$, $j = 1,\dots,m$ may be positive, zero or negative depending on whether $\overline{z}_j \overset{<}{\underset{>}{=}} z_j(w,r,y)$.

[14] Clearly (U^{f}) and (U^{s}) are special cases of (H) and $(\tilde{H})$ with linear constraints. There is, however, a new feature in (U^{s}) : some choice variables become parameters.

When however, we examine **the interrelations** between (U^f) and (U^s), it is quickly realized that the Envelope theorem has a far greater scope than our analysis in § 3 may have led us to expect.

Indeed, **four Le Chatelier effects** can be distinguished between (U^f) and (U^s). As the following diagram illustrates[15], in panel **(i)** our consumer has solved (U^f) at point A. When threatened with quantity rationing, $\bar{z}$, he wishes first to consider the damage inflicted under the best possible circumstances. He thus attains an Envelope tangency at A by choosing $\tilde{z} \equiv z(w,r,y)$ and so he gets $x(w,r,y,\tilde{z}) \equiv x(w,r,y)$, $\lambda(w,r,y,\tilde{z}) \equiv \lambda(w,r,y)$ and $\mu(w,r,y,\tilde{z}) \equiv 0_m$. Thus when w and y vary, while $\tilde{z}$ respond appropriately, we obtain the curvature conditions[16]

$$\left\{\begin{array}{c} \text{The } \ell \times \ell \text{ matrix} \\ [\tilde{X}^s_w + \tilde{x}^s_y \tilde{x}^{s\prime}] - [X^f_w + x^f_y x^{f\prime}] \\ \text{is negative semi-definite, with} \\ \text{at least negative main diagonal} \\ \text{elements} \end{array}\right\} \tag{36}$$

This is, of course, **Samuelson**'s (1947) original **Le Chatelier Principle**, as has been applied to the theory of consumer choice by **Tobin** and **Houthakker** (1950) and **Pollak** (1969). We also note that no Envelope problem is needed. Similarly, in panel **(ii)**, where (U^s) has been solved, at point A, our consumer may wish to ascertain how much better off he would have been if he were forced there while solving (U^f), under appropriate "shadow" prices of rationed goods $\tilde{r}$ and income, $\tilde{y} = y + (\tilde{r} - r)'\bar{z}$. In fact this can be done if he chooses $\tilde{r} \equiv (1/\lambda(w,r,y,\bar{z})\ f_z\,(x(w,r,y,\bar{z}),\bar{z}) \equiv r + (1/\lambda(w,r,y,\bar{z}))\ \mu(w,r,y,\bar{z}) > 0_m$. Then he would attain an Envelope tangency at A with $x(w,\tilde{r},\tilde{y}) \equiv x(w,r,y,\bar{z})$, $z(w,\tilde{r},\tilde{y}) \equiv \bar{z}$ and $\lambda(w,\tilde{r},\tilde{y}) \equiv \lambda(w,r,y,\bar{z})$. Thus when w and y vary, while $\tilde{r}$ and $\tilde{y}$ respond appropriately, we obtain the curvature conditions

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[15] The diagram serves its purpose, despite its simplicity, with $\ell = m = 1$ and $n = 2$.

[16] An ~ superscript indicates $x(w,r,y,\tilde{z})$ and its rates of change in w and y.

$$\left\{\begin{array}{c}\text{The } \ell \times \ell \text{ matrix}\\ [X^s_w + x^s_y x^{s\prime}] - [\tilde{X}^f_w + \tilde{x}^f_y \tilde{x}^{f\prime}]\\ \text{is negative semi-definite, with}\\ \text{at least negative main diagonal}\\ \text{elements}\end{array}\right\} \qquad (37)$$

But that is not all! In panel **(iii)** our consumer has solved (U^f) at point A, but is forced to consume $\overline{z} \neq z(w,r,y)$ in the second-best solution at point B. He can then solve the Envelope problem

$$\max_{\hat{r}}\{v(w,r,y,\overline{z}) - v(w,\hat{r},y+(\hat{r}-r)'\overline{z})\} \qquad (E_{\hat{r}})$$

and determine the shadow price vector, $\tilde{r}$, implicitly from $\overline{z} \equiv z(w,\tilde{r},y+(\tilde{r}-r)'\overline{z})$ and $\tilde{y} \equiv y+(\tilde{r}-r)'\overline{z}$ [17]. Thus the consumer attains an Envelope tangency at the second - best (U^s) at point B, with $x(w,r,y,\overline{z}) \equiv x(w,\tilde{r},y+(\tilde{r}-r)'\overline{z})$, $\overline{z} \equiv z(w,\tilde{r},y+(\tilde{r}-r)'\overline{z})$ and $\lambda(w,r,y,\overline{z}) \equiv \lambda(w,\tilde{r},y+(\tilde{r}-r)'\overline{z})$. Thus as w and y vary, while $\tilde{r}$ and $\tilde{y}$ respond, we obtain the same curvature conditions as in (37), despite the completely different mechanism that generates them.

Finally, in panel **(iv)**, a consumer who has solved (U^s) at point A may still be interested in solving the Envelope problem

$$\max_{\overline{z}}\{v(w,r,y,\overline{z}) - v(w,r,y)\} \qquad (E_{\overline{z}})$$

and derive $\tilde{z}$ implicitly from $v_{\overline{z}}(w,r,y,\tilde{z}) \equiv \mu(w,r,y,\tilde{z}) \equiv 0_m$ [18]. If $\mu(w,r,y,\overline{z}) \neq 0_m$, then clearly $\overline{z} \neq \tilde{z}$ and point B is different from A. With $x(w,r,y) \equiv x(w,r,y,\tilde{z})$, $z(w,r,y) \equiv \tilde{z}$ and $\lambda(w,r,y) \equiv \lambda(w,r,y,\tilde{z})$, when w and y vary, while $\tilde{z}$ responds, we attain an Envelope tangency at the first-best maximum, at B, with curvature conditions which appear as in (36) above.

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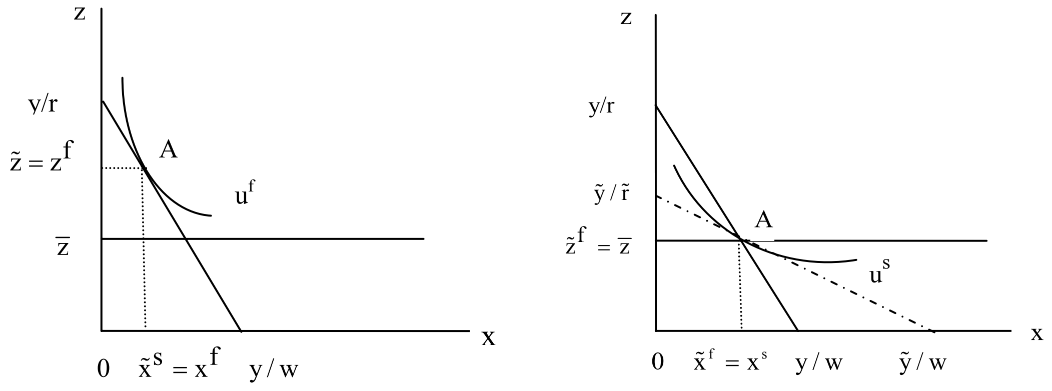
[17] As shown in **Drandakis** (2007), a local maximum of zero is attained in $(E_{\tilde{r}})$ by the above choice of $\tilde{r}$ and $\tilde{y}$. Indeed $v(w,r,y,\overline{z}) = \min_{\hat{r}}\{v(w,\hat{r},y,+(\hat{r}-r)'\overline{z})\} = v(w,\tilde{r},y+(\tilde{r}-r)'\overline{z})$.

[18] Again a local maximum of zero is attained in $(E_{\overline{z}})$ by the choice of $\tilde{z}$. Here $v(w,r,y,) = \max_{\overline{z}}\{v(w,r,y,\overline{z})\} = v(w,r,y,\tilde{z})$.

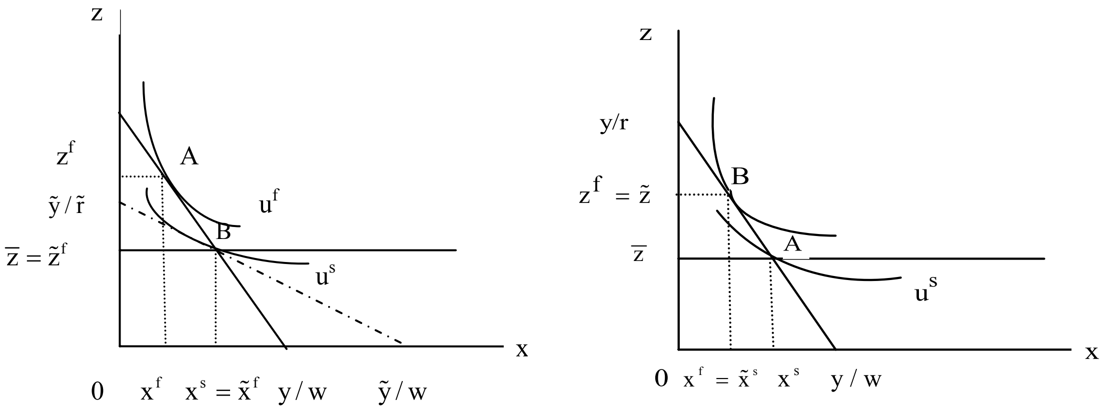


(U^f) at A $\rightarrow (\tilde{U}^s)$ at A

panel (i)

(U^s) at A $\rightarrow (\tilde{U}^f)$ at A

panel (ii)



(U^f) at $A + (E_{\tilde{r}}) \rightarrow (\tilde{U}^s)$ at B

panel (iii)

(U^s) at $A + (E_{\bar{z}}) \rightarrow (\tilde{U}^f)$ at B

panel (iv)

Figure

Panels (i), (ii), (iii) and (iv). Initial utility maxima at A and Envelope tangencies at A or B.

6. Concluding Remarks

The dual method for sensitivity analysis and Le Chatelier Principle, in general static optimization problems, has been presented here. The method is based on the curvature conditions derived from appropriate Envelope problems, in which former parameters become the choice variables and vice versa.

These envelope problems may be either constrained or unconstrained – depending on the parametric structure of the original problem – and may lead to second-order conditions that are subject to constraints. In that event, a further effort is needed for deriving their representation in tangent subspaces and thus producing the suitable curvature conditions. An important role in this process is played by second-order derivative properties of the maximum value functions of the optimization problems.

Despite the early appearance of the two basic papers of **Silberberg** (1974) and **Hatta** (1986), the development of a unified dual method via the Envelope theorem has been undoubtedly delayed by the almost exclusive attention given in the literature to first-order envelope tangencies and the rate mention of the fact that the desired curvature conditions are derived from second-order envelope conditions. Indeed, it is quite clear that the “primal-dual method” of **Silberberg,** i.e.,

$$\min_{\alpha}\{d(\alpha,x^0)\} \equiv \min_{\alpha}\{\varphi(\alpha)-f(x^0,\alpha)\ \ |h(x^0,\alpha)'=0'_m\} \qquad (PD_S)$$

is exactly our first envelope problem for (S), while the “gair function method” of **Hatta**, i.e.,

$$\min_{\alpha}\{g(\alpha,x^0)\} \equiv \min_{\alpha}\{\varphi(\alpha,h(x^0,\alpha))-f(x^0,\alpha)\} \qquad (G_H)$$

is the unconstrained version of our first envelope theorem for (H). Yet, **Silberberg** (1974), pg. 162, considers the f-o-c of (E_S^c) as tantamount to “the famous envelope theorem developed by **Samuelson**”, while **Hatta** (1980), pg. 900, calls his theorem 4 - about $f_\alpha(x(\alpha,\gamma),\alpha)'\ \{X_\alpha(\alpha,\gamma)+X_\gamma(\alpha,\gamma)\ H_\alpha(x(\alpha,\gamma),\alpha)\}=0'_r$ - as “the first envelope theorem” and his theorem 5 on $\varphi_\alpha(\alpha,\gamma)+H_\alpha(x(\alpha,\gamma),\alpha)'\varphi_\gamma(\alpha,\gamma)=f_\alpha(x(\alpha,\gamma),\alpha)$ as “the second envelope theorem”.

In a more recent contribution, **Caputo** (1996) examines the **Hatta** model and, in his main theorem, he proves that the unconstrained gain optimization problem (G_H) and the constrained primal – dual optimization problem for (H), i.e.,

$$\min_{\alpha,\gamma}\{d(\alpha,\gamma,x^0)\}\equiv\min_{\alpha,\gamma}\{\varphi(\alpha,\gamma)-f(x^0,\alpha)\ \mid h(x^0,\alpha)'=\gamma'\} \qquad (PD_H)$$

lead to the same curvature conditions. Indeed, **Caputo** has proved the constrained version of the first envelope problem for (H), (E_H^c) of section 2(f). **Caputo** does not attempt to derive the appropriate curvature conditions for (S), although he notes that the primal - dual method can be applied to the more general **Silberberg** model[19].

We may wonder, therefore, despite the claim of **Hatta** (1987), pg. 157, that several authors "have simplified the proof methods of the Le Chatelier Principle by regarding the principle as an envelope result, following **Mc Kenzie's** (1957) proof of the negative semidefiniteness of the substitution matrix", whether the impact of the Envelope theorem for sensitivity analysis and Le Chatelier Principle was fully appreciated in the relevant literature and whether it were exploited with commendable rapidity.

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[19] **Caputo**, too, notes in pg. 244 that "the first-order necessary conditions [$g_\alpha(\alpha^0, x^0) = 0_r$] yield a one line proof of the envelope theorem for [(H)]".

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Appendix A:

In § 2(c), the product matrix $Z^{0\prime}C^0Z^0 = [I_{rr}, -H_\alpha^{0\prime}H_\gamma^{0^{-1\prime}}]$

$$\begin{bmatrix} -[F_{\alpha x}^0 - H_{\alpha x}^0]X_\alpha^0 + H_\alpha^{0\prime}\Lambda_\alpha^0 & , & -[F_{\alpha x}^0 - H_{\alpha x}^0]X_\gamma^0 + H_\alpha^{0\prime}\Lambda_\gamma^0 \\ -[F_{\gamma x}^0 - H_{\gamma x}^0]X_\alpha^0 + H_\gamma^{0\prime}\Lambda_\alpha^0 & , & -[F_{\gamma x}^0 - H_{\gamma x}^0]X_\gamma^0 + H_\gamma^{0\prime}\Lambda_\gamma^0 \end{bmatrix} \begin{bmatrix} I_{rr} \\ -H_\gamma^{0^{-1}}H_\alpha^0 \end{bmatrix} =$$

$$= \{[-[F_{\alpha x}^0 - H_{\alpha x}^0] + H_\alpha^{0\prime}H_\gamma^{0^{-1\prime}}[F_{\gamma x}^0 - H_{\gamma x}^0]]X_\alpha^0, [-[F_{\alpha x}^0 - H_{\alpha x}^0] + H_\alpha^{0\prime}H_\gamma^{0^{-1\prime}}[F_{\gamma x}^0 - H_{\gamma x}^0]]X_\gamma^0]\}$$

$$\begin{bmatrix} I_{rr} \\ -H_\gamma^{0^{-1}}H_\alpha^0 \end{bmatrix} = [-[F_{\alpha x}^0 - H_{\alpha x}^0] + H_\alpha^{0\prime}H_\gamma^{0^{-1\prime}}[F_{\gamma x}^0 - H_{\gamma x}^0]][X_\alpha^0 - X_\gamma^0 H_\gamma^{0^{-1}} H_\alpha^0] \ ,$$

exactly as appears in (8).

Appendix B:

In § 2 (f), we may offer an alternative proof of the unconstrained Envelope problem (E_H), if we follow **Hatta** (1980) and modify (H) itself in an appropriate way. Indeed let us define, for any $\bar{x} \in X$, $x(\alpha;\bar{x}) \equiv x(\alpha, h(\bar{x},\alpha))$ and $\lambda(\alpha;\bar{x}) \equiv \lambda(\alpha, h(\bar{x},\alpha))$ and note that, although $x(\alpha;\bar{x}) \neq \bar{x}$ in general, we get

$X_\alpha(\alpha;\bar{x}) = X_\alpha(\alpha, h(\bar{x},\alpha)) + X_\gamma(\alpha, h(\bar{x},\alpha))H_\alpha(\bar{x},\alpha)$ and

$\Lambda_\alpha(\alpha;\bar{x}) = \Lambda_\alpha(\alpha, h(\bar{x},\alpha)) + \Lambda_\gamma(\alpha, h(\bar{x},\alpha))H_\alpha(\bar{x},\alpha)$ connecting the derivatives of $x(\alpha;\bar{x})$ and $\lambda(\alpha;\bar{x})$ with those of the solution of (H), when constraint levels also vary so as to keep $h(x,\alpha) = h(\bar{x};\alpha)$. We see therefore that we can get an unconstrained Envelope problem, if we consider the **compensated** version of (H), i.e.,

$$\varphi(\alpha;\bar{x}) \equiv \max_x \{f(x,\alpha) \ | h(x,\alpha)' = h(\bar{x},\alpha)'\} \ , \qquad (H_{comp})$$

whose solution $x(\alpha;\bar{x})$ and $\lambda(\alpha;\bar{x})$ satisfy (10) and (11) in the text, appropriately modified.

We also have $\varphi(\alpha;\bar{x}) \equiv f(x(\alpha;\bar{x}),\alpha) \equiv \varphi(\alpha, h(\bar{x},\alpha))$ with

$$\varphi_\alpha(\alpha;\bar{x}) = f_\alpha(x(\alpha;\bar{x}),\alpha) - [H_\alpha(x(\alpha;\bar{x}),\alpha) - H_\alpha(\bar{x},\alpha)]'\lambda(\alpha;\bar{x})$$

and

$$\Phi_{\alpha\alpha}(\alpha;\overline{x}) = [F^c_{\alpha x} - H^c_{\alpha x}]X^c_{\alpha} + [F^c_{\alpha\alpha} - H^c_{\alpha\alpha} + H_{\alpha\alpha}(\overline{x},\lambda^c,\alpha)] - [H^c_{\alpha} - H_{\alpha}(\overline{x},\alpha)]'\Lambda^c_{\alpha} \ .$$

Here superscript c indicates that the respective matrices are evaluated at $x(\alpha;\overline{x})$ and $\lambda(\alpha;\overline{x})$.

Of course, we are not interested in any odd $\overline{x}$ but in studying the impact of variations in α and γ evaluated at a solution of (H). Thus for any (α^0, γ^0) let $x^0 = x\,(\alpha^0, \gamma^0)$ and $\lambda^0 = \lambda\,(\alpha^0, \gamma^0)$ with $x\,(\alpha^0\,;\,x^0) = x^0$ and $\lambda\,(\alpha^0\,;\,x^0) = \lambda^0$. We can then consider the unconstrained Envelope problem

$$\max_{\alpha}\{f(x^0,\alpha) - \varphi(\alpha;x^0)\} \quad , \qquad (E^{comp}_{H})$$

with

$$f-o-c\ \{f_{\alpha}(x^0,\alpha) - \varphi_{\alpha}(\alpha;x^0) = 0_r\}$$

and

$$s-o-c\begin{Bmatrix} [F_{\alpha\alpha}(x^0,\alpha) - \Phi_{\alpha\alpha}(\alpha;x^0)] \text{ is a} \\ \text{negative semidefinite (or definite) matrix} \end{Bmatrix}$$

satisfied at α^0 with $x\,(\alpha^0\,;\,x^0) = x^0$. We also see that $\varphi_{\alpha}\,(\alpha^0\,;\,x^0) = f_{\alpha}\,(x^0\,;\alpha^0)$ and $\Phi_{\alpha\alpha}(\alpha^0;x^0) = [F^{0c}_{\alpha x} - H^{0c}_{ex}]X^{0c}_{\alpha} + F^{0c}_{\alpha\alpha}$ and thus we get the curvature conditions of (20) in the text.

Appendix C:

In § 3, the s-o-c of $(E_{S\tilde{S}})$ involve the κ x κ matrix

$$[F^0_{\alpha x} - H^0_{\alpha x}][\tilde{X}^0_{\alpha} - X^0_{\alpha}] - \tilde{H}^{0\prime}_{\alpha}[\tilde{\Lambda}^0_{\alpha} - \begin{bmatrix} \Lambda^0_{\alpha} \\ O_{m^+m^+} \end{bmatrix}].$$

When α becomes $(\alpha,\tilde{\gamma})$, the above matrix is written as

$$\underset{(r'+m')\,x\,n}{\begin{bmatrix} F^0_{\alpha x} & - & H^0_{\alpha x} \\ F^0_{\tilde{\gamma}x} & - & H^0_{\tilde{\gamma}x} \end{bmatrix}}\underset{n\,x\,(r'+m')}{[\tilde{X}^0_{\alpha} - X^0_{\alpha}, \tilde{X}^0_{\tilde{\gamma}} - X^0_{\gamma}]} - \underset{(r'+m')\,x\,m'}{\begin{bmatrix} \tilde{H}^{0\prime}_{\alpha} \\ H^{0\prime}_{\tilde{\gamma}} \end{bmatrix}}\underset{m'\,x\,(r'+m')}{[\tilde{\Lambda}^0_{\alpha} - \begin{bmatrix} \Lambda^0_{\alpha} \\ O_{m^+m^+} \end{bmatrix}, \tilde{\Lambda}^0_{\gamma} - \begin{bmatrix} \Lambda^0_{\gamma} \\ O_{m^+m^+} \end{bmatrix}]} \ .$$

When this matrix is multiplied on the left by $Z^{0\prime} = [I_{r'r'} \, , -\tilde{H}_{\alpha}^{0\prime}\tilde{H}_{\tilde{\gamma}}^{0-1\prime}]$ it gives us the $r' \times \kappa$ matrix

$$\begin{aligned}
&\{F_{\alpha x}^{0} - H_{\alpha x}^{0} - \tilde{H}_{\alpha}^{0\prime}\,\tilde{H}_{\tilde{\gamma}}^{0^{-1}\prime}[F_{\tilde{\gamma}x}^{0} - H_{\tilde{\gamma}x}^{0}]\}[\tilde{X}_{\alpha}^{0} - X_{\alpha}^{0} \, , \tilde{X}_{\tilde{\gamma}} - X_{\tilde{\gamma}}^{0}] - \\
&-[\tilde{H}_{\alpha}^{0\prime} - \tilde{H}_{\alpha}^{0\prime}\,\tilde{H}_{\tilde{\gamma}}^{0^{-1}\prime}\,\tilde{H}_{\tilde{\gamma}}^{0\prime}][\tilde{\Lambda}_{\alpha}^{0} - \begin{bmatrix} \Lambda_{\alpha}^{0} \\ O_{m^{+}m^{+}} \end{bmatrix}, \; \tilde{\Lambda}_{\tilde{\gamma}}^{0} - \begin{bmatrix} \Lambda_{\gamma}^{0} \\ O_{m^{+}m^{+}} \end{bmatrix}] = \\
&= \{F_{\alpha x}^{0} - H_{\alpha x}^{0} - \tilde{H}_{\alpha}^{0\prime}\,\tilde{H}_{\tilde{\gamma}}^{0^{-1}\prime}[F_{\tilde{\gamma}x}^{0} - H_{\tilde{\gamma}x}^{0}]\}[\tilde{X}_{\alpha}^{0} - X_{\alpha}^{0} \, , \tilde{X}_{\tilde{\gamma}} - X_{\tilde{\gamma}}^{0}] \quad ,
\end{aligned}$$

which, when multiplied on the right by Z^{0}, gives us (26).

Again it should be noted how helpful matrix notation is in carrying out such complex operations.