Simultaneous Online Auctions by Sellers of Different Reputations: Theory and Experimental Evidence

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Abstract

We study a setting where vertically-differentiated sellers (e.g. sellers of different reputations for honesty) simultaneously offer sealed-bid, second-price, single-unit auctions for identical goods to unit-demand buyers. Such vertical differentiation is a salient but understudied aspect of wide range of online auction marketplaces. We characterize the form of the bidding equilibria and derive expressions for the corresponding allocative efficiency and expected seller revenue. When bidders are restricted to submit at most one bid, our theory predicts the existence of a unique Bayes-Nash equilibrium that resembles a form of probabilistic "mating-of-likes". Allowing unit-demand bidders to place an arbitrary number of bids induces complex mixed strategy profiles where bidders place positive bids in all available auctions. The probabilistic nature of the bidding equilibria introduces allocative inefficiencies that arise from the lack of coordination in auction selection among bidders. We test our theoretical propositions in a controlled laboratory experiment with induced values. Our experimental analysis finds significant conformance with, as well as some departures from, theory. We find evidence of a probabilistic assortative matching, in that higher bidder types are more likely to choose higher reputation sellers, but they do so in a randomized manner. As predicted, and in contrast to the single auction setting, we find evidence of convex returns to reputation to sellers. We find that seeking safety, bidders tend to crowd on the highest reputation seller. They also fail to appropriately adjust for risk associated with the lowest reputation seller. Such departures from efficient bidding further exacerbate the inefficiency of simultaneous auctions, hurting bidders but benefiting extreme reputation sellers.

Keywords: Simultaneous online auctions, reputation, equilibrium bidding strategy, laboratory experiment

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1. Introduction and Motivation

While much of auction theory and online auction research has taken as its unit of analysis a single auction, large-scale decentralized online marketplaces, such as eBay, typically consist of multiple competing auctions offering identical or very similar goods very often sold by sellers of different reputations. Most online markets carry an element of risk as there is a possibility that a seller may receive payment but not deliver the promised goods. To mitigate this risk, most marketplaces implement some notion of *seller reputation,* serving as an estimate of the probability that a seller will deliver on his promises, but also introducing a dimension of *vertical differentiation* among sellers that complicates bidding. As an illustration of this fact, if, on September 1, 2008, an aspiring iPod owner searched eBay for an "iPod Nano 4GB," she would have been confronted with more than a thousand almost identical listings sold by sellers of widely varying eBay "feedback scores". Many of these listings end within seconds of each other, rendering them practically simultaneous. Our hypothetical bidder only needed one iPod, raising the question as to whose auction(s) should she place her bid(s) and in what amount(s)? Using a combination of theory and controlled experiments, this work aims to provide rigorous answers to a stylized version of this common, but surprisingly under-researched phenomenon.

Our theoretical analysis derives the form of equilibrium bidding strategies in settings where unit-demand bidders face simultaneous auctions of identical goods by sellers with different reputations. Our analysis also derives predictions about the allocative efficiency of simultaneous auctions and the impact of reputation on seller revenues in such settings. We design an induced value (Smith 1976) based controlled laboratory experiment that tests our theory, providing us with insights about actual bidder strategies and the degree of conformance to the predicted equilibria. Where we observe departures from theory, we conjecture plausible explanations and motivate future work to capture fully the behavioral complexities of such market environments.

The study of simultaneous auctions is of growing practical importance, with a scope that goes well beyond online auction houses such as eBay. Ausubel and Cramton (2008) showcase the usage of simultaneous clock auctions for electricity, natural gas, telecom spectrum, emissions trading and the sale of uncut diamonds. While the examples highlighted by Ausubel and Cramton (2008) are run on behalf of a single entity (for example, a national telecom authority), we consider a more general setting where multiple sellers, with varying reputations, are engaging in simultaneous auctions. This brings in an element of vertical differentiation within the set of simultaneous auctions. While we primarily consider the source of this differentiation to be seller reputation, it is important to note that much of our analysis of *identical goods* and *different reputations* is also valid in settings where vertically-differentiated *imperfect substitute goods* are sold by the *same* seller. On eBay this could be reflected in a merchant seller simultaneously selling multiple grades of used iPods such that auction listings are not truly identical. In the case of mineral rights (oil, timber) auctions such a setting would arise if multiple adjacent tracts that cannot be expected to be perfect substitutes of one another were being auctioned simultaneously. These examples suggest that there is an evident trend towards vertically differentiated simultaneous auctions and therefore studying them both theoretically and experimentally is of growing practical importance.

Our game-theoretic analysis of the equilibrium bidding strategies is conducted in two stages. In stage one, in the interest of analytical tractability and with a desire to obtain closed form expressions for quantities of interest, we restrict bidders to submitting at most one bid. In stage two we relax this assumption and allow for the case where bidders may submit multiple bids. When bidders are restricted to submit at most one bid, there exists a unique Bayes-Nash equilibrium that resembles a stochastic version of the mating-of-likes equilibrium (Becker 1973): Sellers are ranked according to their reputation and buyers self-select into a finite number of zones (the boundaries of which we are able to derive in closed form) according to their types, as reflected in their private valuations for the good. Buyers whose types fall in the highest zone always bid on the highest seller; buyers whose types fall in the 2nd highest zone randomize between the top two sellers, assigning higher probability to selecting the 2nd seller. More generally, buyers whose types fall in the *k*-th zone randomize between the top *k* sellers, assigning increasingly higher probability to selecting less reputable sellers. This bidding behavior is a form of probabilistic positive assortative matching: bidders assess where they stand on the valuation scale and assign higher probability to bidding on the auction that ``matches'' their respective zone, while also occasionally bidding on ``higher'' auctions. When the single bid assumption is relaxed, we show that, even though bidders have unit demand, in equilibrium it is optimal for bidders to place non-zero bids in all auctions. The optimal bid amount in each auction is equal to the bidder's expected valuation of the respective auction (taking into account the seller's reputation) multiplied by the probability that the bidder will not receive the item from any of the other auctions, given her other bids. This specification gives rise to complex mixed equilibria which, however, maintain the mating-of-likes property of the single-bid case: higher bidder types have a higher probability of placing a bid close to their expected valuation on a high reputation seller.

In order to test our predictions, we follow a long tradition of controlled laboratory experiments (Smith 1976) using induced value theory to control for heterogeneous private values. We model seller reputation based on prior experimental work (Bolton et al. 2004) as the probability of a seller shipping an item. We impose subject private values exogenously, allowing us to study the decision making behavior of bidders in a controlled setting without the confounding effects of subjective bidder types. Our experimental analysis finds interesting conformance as well as some departures from theory. We find support for our probabilistic mating-of-likes equilibrium. We find seller selection by bidders is monotone positive in bidder type as higher bidder types are more likely to choose higher reputation sellers, and lower types are seen targeting the lower reputation sellers. Interestingly, the probabilistic aspect of the mating-of-likes is evident in that the range of sellers targeted by the bidders. This range is inversely related to the buyer's type zone, with the highest type zone buyers restricting their attention to the top reputation seller and the low buyer type zone randomizing across all sellers. We observe that when given the opportunity to place bids with as many as four sellers, 78% of bidders resort to bidding on a single auction and (with only one exception) the remaining bidders place at most two bids. Consistent with our theory, bidders who bid on two auctions tend to place one serious bid and another lowball bid. However, they appear to do this in a manner that results in a crowding of bidders on the highest and lowest reputation sellers. Bidders targeting lower reputation (higher risk) sellers generally fail to appropriately adjust for the associated risk. Taken together the last two effects result in economic rents for the highest and lowest reputation sellers and lower bidder surplus than theory would predict.

We show that simultaneous auctions suffer from low allocative efficiency arising from the lack of coordination in auction selection among bidders. This is an inherent drawback of decentralized auctions with incomplete information that has particularly severe consequences in settings with vertical differentiation. The intuition behind this comes from the probabilistic nature of our bidding equilibria: In the absence of complete information about each other's private value, bidders cannot coordinate with each other when choosing where to bid. It is then possible that two or more high value bidders who could all win if they could coordinate and choose adjacent auctions will cluster on the same auction, reducing efficiency.

The rest of the paper is organized as follows. Section 2 reviews the related literature. Section 3 introduces our theoretical model. Section 4 describes the setup and results from our controlled laboratory experiment. Finally, Section 5 concludes and discusses possible directions of future work.

2. Literature Review

The study of competing auctions is the focus of a growing literature in economics and management science. For the benefit of the reader and to more precisely position our work, we structure this literature along three dimensions. First, we look at the temporal nature of competition among auctions. We distinguish between what the literature has to say about simultaneous, sequential and overlapping auctions. Second, we classify past work with respect to whether the competing auctions are assumed to be identical or whether they are vertically differentiated imperfect substitutes. Third, we look at the information environment of competing auctions, differentiating studies where bidders are assumed to have complete vs. incomplete information about each others' valuation.

There are three categories of competing auctions: simultaneous auctions, sequential auctions and overlapping auctions. The main sequential auction result (Ashenfelter 1989) in the literature is that prices (of French wines and Australian wool auctions) are observed to decline in subsequent auctions. DeSilva et al. (2005) find that in sequential auctions of synergistic tasks, winners in the earlier auctions are more likely to participate in later auctions. Moreover, conditional on participation, past winners place lower bids, on average, and are so more likely to win in later auctions. Goes et al (2009) model the willingness to pay formation of repeat bidders in sequential online auctions. The key simultaneous ascending auction result (Peters and Severinov 2006) is that when there are no bidding costs and when bidders share a common fixed buying horizon (or ending time), it is an equilibrium strategy for bidders to submit a bid on an auction with the lowest standing bid. If the strategy is followed by all the bidders, prices are expected to be uniform across all auctions. In the context of identical goods, Krishna and Rosenthal (1996) study the revenue effects of simultaneous sealed-bid, second-price auctions and compare effects of single unit-demand versus multi-unit demand buyers. Similar to our setting, they assume bidder values are independent and private, and bidders only know the distribution of values but have no information about the valuation of others. Rosenthal and Wang (1996) extend the analysis to settings where buyers have common values and overlapping interests. In an empirical study of Sam's Club's online auctions, Bapna et al. (2008) introduce the notion of an overlapping online auction market to characterize a setting where there are multiple ongoing auctions for the identical item and these auctions share a non-zero time interval with each other. Overlapping auctions subsume simultaneous auctions as a special case where opening and closing times are identical. Bapna et al. (2008) find that overlapping auctions attract 'institutional' bidders, who bid in a participatory manner across multiple auctions, and that such bidders exert a downward pressure on auction prices. They also find that overlap of an auction with other competing auctions has a significant negative influence on unit prices.

However, extant research has yet to theoretically characterize the equilibrium biddings strategies for overlapping auctions. We focus on simultaneous auctions and expand the literature in this area by considering a setting where auctions are not identical and are (vertically) differentiated by the seller's reputation.

Another stream of competing auctions literature deals with varying information environments or by considering a more restrictive single-item or two, and, in one case, three auctions setting. Peters and Severinov (1997) study simultaneous auctions of identical goods where sellers compete by setting different reserve prices. They manipulate the information buyers have about their private values and find this information has an effect on auction outcomes. Bikhchandani (1999) studies simultaneous auctions of heterogeneous objects to multi-unit demand buyers where valuations are commonly known and correspond to other goods the buyer obtains. Szentes (2007) studies two-object, two-bidder simultaneous auctions in settings where information is complete and the objects are either complements or substitutes. Byde (2001) and Bertsimas, Hawkins and Perakis (2009) propose dynamic programming bidding algorithms for a single item in an online auction, as well as for multiple items in three simultaneous or overlapping online auctions. They however “ignore the effect” (page 23) of ratings of sellers on bidder behavior. Beil and Wein (2008) study a setting of two simultaneous auctions for identical goods where some of the unit-demand buyers are dedicated to each auction while others participate in both auctions.

On the reputation front, analytical and experimental work documenting the effects of reputation on economic decisions includes that of Kreps and Wilson (1982), Berg, Dickaut and McCabe (1995), Lucking-Reiley (2000) and Bolton et al. (2004). The vast majority of the online auction reputation literature is based on observational empirical settings that use some variant of hedonic regression techniques to tease out the impact of reputation on variables of interest. For a more detailed review of this please refer to Dellarocas (2003) and Fan et al. (2005). The main finding to date has been the existence of linear rents to reputation for sellers. However, the impact of differing seller reputation in a simultaneous auction setting, the likely default setting facing most consumers on sites such as eBay, has not yet been studied. We are motivated to ask whether reputation rents can be expected to be linear in this more complex environment, or should sellers of higher reputation benefit even more from potential coordination failures in this environment. Consider, for instance, Peters and Severinov's (2006) prediction that prices are expected to be uniform across competing auctions. We ask whether this holds under vertically differentiated imperfect substitutability arising from reputational difference of sellers.

We do this by deriving equilibrium bidding strategies in simultaneous auctions of identical goods by sellers of different reputations and testing the predictions of this theory in a controlled laboratory setting with economic incentives (Smith 1976). Much like Resnick et al. (2006), our motivation for using the controlled laboratory setting is to avoid possible endogeneity from the presence of other unobserved covariates that impact reputation and auction outcomes.

In summary, while our study draws on the above streams of literature, we believe that ours is the first paper that looks at *simultaneous*, *vertically-differentiated* auctions where bidders have *incomplete information* about everyone else's *private* value. It is also the first paper that derives theoretical predictions about the *impact of a seller's reputation in a competing auctions environment*. We next develop our theoretical model.

3. Theoretical Model

Our model considers M sellers and N risk-neutral buyers trading in a market. Each seller offers a sealed-bid, second-price auction for a single unit. Items are identical across sellers and we assume that each buyer has an inelastic demand for one unit of this good. All seller auctions begin and end simultaneously. Each buyer has a privately known type $t \in [0,1]$ that determines her unit valuation. Buyer t's valuation of seller k's good is equal to $v_k(t) = tw$, Buyer types are independently drawn from the same cumulative probability distribution $F(t)$ with associated density function $f(t) > 0$. Buyers know the total number of buyers N, their own type t, and the probability distribution of other buyers' types.

Each seller k, $1 \leq k \leq M$ has an associated reputation $0 \leq r_k \leq 1$, interpreted as the buyers' commonly held subjective belief that the seller will fulfill the transaction and will deliver the promised good. Reputations can arise from public information about a seller's past behavior provided by an online reputation mechanism (such as the one used by eBay) or from some other information that is available to all buyers.[1] A risk-neutral buyer of type t who wins seller k's auction at price p thus expects to get surplus $tr_kw - p$.

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[1] We do not make any assumptions with respect to whether such information is accurate or not. We simply assume that it exists and that it is commonly interpreted by all buyers. Readers interested in questions that pertain to the design of reliable online reputation mechanisms are referred to Dellarocas (2003).

In the rest of the paper we assume that a seller's index *k* indicates his relative reputation rank within the set of competing sellers. Specifically, we assume that $r_1 \geq r_2 \geq ... \geq r_M$.

Trade is organized in the following way. All buyers simultaneously arrive in the market and discover their types. Sellers simultaneously announce their auctions. Buyers look up each seller's reputation, then submit bids to a subset of available auctions. Each auction is won by its highest bidder who pays an amount equal to the second-highest bid.

A buyer's decision problem has two distinct components: (i) select *which* auctions to bid on, (ii) decide *how much* to bid on each selected auction.

3.1 One-bid Equilibria

In order to derive our initial insights we consider the special case where each unit-demand buyer is restricted to place at most one bid in competing simultaneous auctions of identical goods by sellers of different reputations. Equilibrium bidding behavior has an elegant characterization in the special case.

Since all auctions are sealed-bid, once buyers have decided on which auction to place their bid, individual auctions proceed independently. According to the theory of second price auctions, buyer *t*'s optimal bid on seller *k*'s auction is independent and equal to her expected valuation $v_k(t) = tr_k$. This observation simplifies our problem considerably as it allows a buyer's bidding strategy to be uniquely determined by her auction selection strategy.

An *auction selection strategy* can be represented as a vector $\mathbf{s}(t) = (s_1(t),..,s_M(t)) \in [0,1]^M$ where $s_k(t)$ denotes the probability that buyer *t* will bid (an amount equal to tr_k) on seller *k*'s auction. An auction selection strategy is *pure* if and only if all components $s_k(t)$ are either 0 or 1. Throughout this section we restrict our attention to strategy vectors that are piecewise continuous in buyer type *t*.

If buyers are restricted to select at most one auction, they will choose the auction that maximizes their expected surplus, given their beliefs about every other buyer's strategy. Specifically, buyer *t*'s expected surplus from bidding her expected valuation on seller *k*'s auction is equal to:

$$V_k(t) = \sum_{n=0}^{N-1} Pr[n \text{ other bidders choose auction } k] \times Pr[\text{all } n \text{ other bidders have types} \leq t] \\ \times r_k w(t - E[\text{2nd highest type} \mid \text{highest type} = t \wedge \exists n \text{ other bidders}]) \tag{1}$$

Observe that the expected surplus from winning an auction increases with a seller's reputation but also with the expected distance between the highest and second-highest bidders' types. Auctions by less reputable sellers that receive fewer bids may, thus, generate a higher expected surplus than auctions by more reputable sellers that receive more bids. Furthermore, the probability of winning an auction decreases with the number of bidders of similar or higher types. In selecting their strategy, buyers must, therefore, trade off the incentive to bid on reputable sellers (increases the probability that the buyer will receive the promised good) against the incentive to bid on less popular auctions (increases the probability of winning and decreases the expected payment). The tension between these two opposing forces gives rise to the resulting equilibria.

Given a selection strategy $\mathbf{s}:[0,1] \rightarrow [0,1]^M$, the following set of functions will play an important role in the analysis:

$$Q_k(t \mid \mathbf{s}) = \int_0^t s_k(u) f(u) du, \quad k = 1, \ldots, M \tag{2}$$

In the rest of the paper we will omit the dependence on **s** when it is implied. The quantity $Q_k(t)$ is equal to the probability that a buyer is of type *t* or lower *and* bids on seller *k*'s auction. The quantity $Q_k(1)$ then represents the probability that a randomly chosen buyer bids on seller *k*'s auction. The quantity $Q_k(1) - Q_k(t) = \int_t^1 s_k(u) f(u) du$ is equal to the probability that a buyer is of type *t* or higher *and* bids on seller *k*'s auction. Finally, the quantity $1 - Q_k(1) + Q_k(t)$ is equal to the probability that a buyer is of type *t* or lower *or* does not bid on seller *k*'s auction. This, in turn, is equal to the probability that a bidder of type *t* will win auction *k* if he is competing against exactly one other bidder.

The following Lemma is a generalization of standard auction theory results (McAfee and McMillan 1987; Riley and Samuelson 1981) in the context of simultaneous auctions:[2]

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[2] Full proofs of all propositions and lemmas are given in the Appendix.

Lemma 1: *Let* $\mathbf{s}(t) = (s_1(t),..,s_M(t)) \in [0,1]^M$ *denote a buyer's beliefs about every other buyer's auction selection strategy and let* $Q_k(t \mid \mathbf{s})$ *be the functions defined by those beliefs and equation (2). The following are true:*

1. *The number of bidders on seller k's auction follows a binomial subjective probability distribution with mass function:*

$$P_k(m \mid \mathbf{s}) = \binom{N}{m} Q_k(1 \mid \mathbf{s})^m \left(1 - Q_k(1 \mid \mathbf{s})\right)^{N-m} \qquad (3)$$

2. *The subjective probability that a buyer of type t who bids her expected valuation on seller k's auction will win the auction is given by:*

$$W_k(t \mid \mathbf{s}) = \left(1 - Q_k(1 \mid \mathbf{s}) + Q_k(t \mid \mathbf{s})\right)^{N-1} \qquad (4)$$

3. *Buyer t's expected surplus from bidding her expected valuation on seller k's auction is given by:*

$$V_k(t \mid \mathbf{s}) = r_k \int_0^t \left(1 - Q_k(1 \mid \mathbf{s}) + Q_k(x \mid \mathbf{s})\right)^{N-1} dx \qquad (5)$$

In the above expressions, the quantity $\left(1 - Q_k(1 \mid \mathbf{s}) + Q_k(t \mid \mathbf{s})\right)^{N-1}$ is equal to the probability that no other bidder has type higher than *t and* bids on seller *k*'s auction.

A (*Bayes-Nash*) symmetric *auction selection equilibrium* is a strategy $\mathbf{s}^* : [0,1] \to [0,1]^M$ that maximizes the expected surplus of all buyer types subject to the assumption that all buyers believe that other buyers are also following strategy $\mathbf{s}^*$. Specifically, $\mathbf{s}^*$ satisfies the following incentive compatibility constraints:

$$\begin{array}{c} s_k^*(t) = 0 \Rightarrow V_k(t \mid \mathbf{s}^*) \le V_\ell(t \mid \mathbf{s}^*) \\ 0 < s_k^*(t) < 1,\, 0 < s_\ell^*(t) < 1 \Rightarrow V_k(t \mid \mathbf{s}^*) = V_\ell(t \mid \mathbf{s}^*) \\ s_k^*(t) = 1 \Rightarrow V_k(t \mid \mathbf{s}^*) \ge V_\ell(t \mid \mathbf{s}^*) \end{array} \qquad \text{for all } t \in [0,1] \text{ and } 1 \le k, \ell \le M \qquad (6)$$

The one-bid restriction requires that $\sum_{k=1}^{M} s_k^*(t) = 1 \quad \text{for all } t \in [0,1]$.

Intuition suggests that bidders whose types are near the top of the distribution (and who, therefore, have a good chance of winning whichever auction they decide to bid on) will select sellers with higher reputations, whereas bidders with low types will choose sellers with lower reputations in the hope of maximizing their prospects of winning something. Interestingly, it turns out that no pure strategy auction selection equilibrium exists.

Proposition 1: *There exists no pure strategy one-bid auction selection equilibrium.*

An informal justification of the non-existence of a pure equilibrium can be based on the following argument. Suppose that a pure strategy equilibrium exists. Then, the assumption of piecewise continuous strategies implies that this equilibrium can be defined in terms of type zones $(\underline{t_k}, \overline{t_k}]$ such that, all types within a zone always bid on a specific seller's auction. Assume a pure strategy that prescribes that all buyers in zone $(0,t]$ bid on seller *k*'s auction. Sellers whose types are near zero have the lowest probability of winning. If low types find it optimal to bid on seller *k*'s auction then higher types will find it even more so. Therefore, any pure equilibrium must have $t=1$: all buyers bid on seller *k*'s auction. But then, any buyer who bids on another seller's auction, is guaranteed to win that auction. If there exists some other seller with positive reputation then buyers whose types are sufficiently close to the bottom of the distribution will prefer to deviate from the pure equilibrium. Thus, no pure strategy can be an equilibrium.

The following proposition sheds further light into the structure of the mixed auction selection equilibria:

Proposition 2: *All one-bid auction selection equilibria satisfy the following properties:*

1. $s_k^*(t_0)=1$ *implies* $s_k^*(t)=1$ *for all* $t \in (t_0,1]$
2. $s_k^*(t_0)=0$ *implies* $s_k^*(t)=0$ *for all* $t \in (t_0,1]$
3. $s_k^*(0)=0$ *implies* $s_\ell^*(0)=0$ *for all* $\ell > k$
4. *For all* $1 \le k, \ell \le M$ *, if there exists* $t_0 \ge 0$ *such that* $s_k^*(t), s_\ell^*(t)$ *both switch from positive values to zero at* t_0 *then* $r_k = r_\ell$ *.*

Propositions 1 and 2 together with the assumption of piecewise continuous strategies allow us to provide a general characterization of auction selection equilibria:

1. All one-bid auction selection equilibria must be mixed everywhere, except for an interval at the top of the type space.
2. Such equilibria can be characterized in terms of a sequence of type intervals:

$$(0,t_{L-1}],(t_{L-1},t_{L-2}],..,(t_z,t_{z-1}],...,(t_1,1]$$

with the property that buyers whose types fall within interval $(t_z, t_{z-1}]$ randomize among the *z* sellers with the highest reputation.

3. The choice set (set of sellers among which buyers randomize) of a given type interval is equal to the choice set of the immediately preceding type interval minus the least reputable seller of that interval.

Figure 1 depicts the general form of these equilibria.

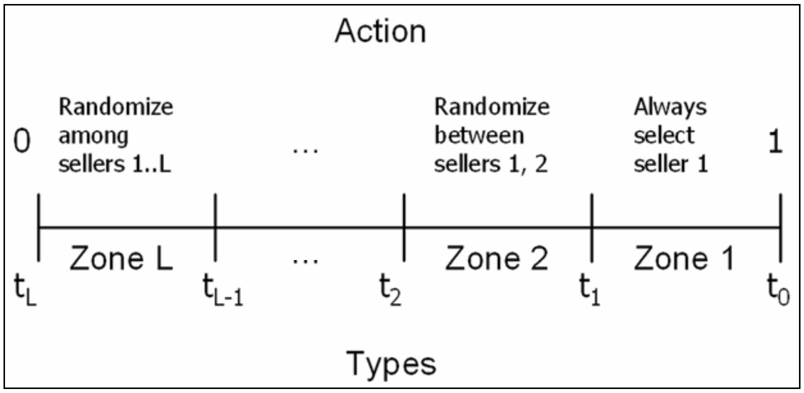

Figure 1. Bidders' Equilibrium Strategy Resembles Probabilistic Assortative Matching

We show that for a given set of seller reputations, there exists a unique one-bid auction selection equilibrium with the above properties. Proposition 3 provides the details:

Proposition 3: *Consider a setting where* $M \geq 2$ *sellers with reputations* $r_1 \geq r_2 \geq ... \geq r_M$ *simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to* $N \geq 2$ *unit-demand buyers, independently drawn from the same type distribution* $F(t)$. *Define* L *as the lowest integer* $2 \leq L \leq M-1$ *for which* $r_{L+1} < \left(\frac{L-1}{\sum_{i=1}^{L} \sqrt[N-1]{\frac{1}{r_i}}} \right)^{N-1}$. *If no such integer exists then* $L = M$. *The following clauses describe the properties of the unique one-bid auction selection equilibrium.*

1. *Buyers are divided into* L *zones according to their type. Let* t_z, $z = 0,1,..,L$, $t_0 = 1, t_L = 0$ *denote the zone delimiters. Buyers whose types satisfy* $t_z < t \leq t_{z-1}$ *belong to zone z.*
2. *Zone-z buyers randomly choose among sellers* $k = 1,..,z$ *with corresponding selection probabilities:*

$$s_{zk} = \frac{\sqrt[N-1]{\frac{1}{r_k}}}{\sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}}}$$

3. *If* $L < M$ *then sellers* $L+1,..,M$ *are never chosen by any buyer.*

4. *Zone delimiters* t_z*,* $z = 0,1,..,L-1$ *are solutions of the following equation:*

$$F(t_z) = \left(\sqrt[N-1]{r_{z+1}} \sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}} \right) - (z-1)$$

5. *The expected number of bids on seller k's auction is equal to:*

$$B_k = \begin{cases} N\left(1-(L-1)\frac{\sqrt[N-1]{\frac{1}{r_k}}}{\sum_{i=1}^{L} \sqrt[N-1]{\frac{1}{r_i}}}\right) & \text{if} k \le L \\ 0 & \text{if} k > L \end{cases}$$

Observe that each type zone's auction selection probabilities s_{zk} are inversely proportional to the reputations of the sellers among which buyers randomize. Thus, *less* reputable sellers are chosen more often by buyers of a given type zone. Nevertheless, because sellers of higher reputation are considered by buyers of more type zones, the total expected number of bids B_k on a seller's auction monotonically increases with his reputation. These testable insights form the basis for our controlled laboratory experiment, which we will describe in the next section

The following is a high-level summary of the equilibrium described by Proposition 3: Sellers are ranked according to their reputation. Buyers self-select into a finite number of zones, according to their types (valuations). Buyers whose types fall in the highest zone always bid on the highest seller; buyers whose types fall in the 2nd highest zone randomize between the top two sellers, assigning higher probability to selecting the 2nd seller. More generally, buyers whose types fall in the *k*-th zone randomize between the top *k* sellers, assigning increasingly higher probability to selecting less reputable sellers. This bidding behavior is a form of probabilistic positive assortative matching, somewhat akin to Becker's (1973) famous mating-of-likes marriage equilibrium: bidders assess where they stand on the valuation scale and assign higher probability to bidding on the auction that "matches" their respective zone, while also occasionally taking chances on "higher" auctions. We examine whether this prediction holds in the laboratory.

The above auction selection equilibria *always* involve some mixing: irrespective of the distribution of seller reputations, buyer types $t \in (0, t_1]$, $t_1 = F^{-1}\left(\sqrt[N-1]{r_2/r_1}\right)$ (see Part 4 of Proposition 3) always randomize between *at least* the two most reputable sellers; only buyer types $t \in (t_1, 1]$ follow a pure bidding strategy.

An interesting aspect of the auction selection equilibrium is Part 3 of Proposition 3: that, if $r_{L+1} < \left(\frac{L-1}{\sum_{i=1}^{L} \sqrt[N-1]{\frac{1}{r_i}}} \right)^{N-1}$, then sellers $L+1, \ldots, M$ do not receive any bids. The condition can be equivalently expressed as:

$$r_{L+1} < \left(\frac{L-1}{L} \right)^{N-1} \left(\frac{1}{L} \sum_{i=1}^{L} (r_i)^{-\frac{1}{N-1}} \right)^{-(N-1)}$$

The latter is a condition between seller-$(L+1)$'s reputation and the $\left(-\frac{1}{N-1}\right)$-power mean of all higher-ranked sellers' reputations. The condition is met as long as r_{L+1} is not substantially lower than the average reputation of higher ranked sellers. The intuitive interpretation of this condition, thus, is that if seller reputations are relatively uniformly spread out in the interval $[0,1]$ then all sellers will receive some bids (buyers whose types belong to the bottom zone randomize among all sellers). If, on the other hand, there is a cluster of sellers whose reputations are substantially higher than those of the rest of the sellers, then it is possible that no buyer will place any bids on the less reputable sellers' auctions.

Proposition 3 subsumes settings where all sellers have identical reputation as a special case. In such settings, it is intuitive to show that they will randomly choose one of the M sellers with equal probability $1/M$.

3.2 Allocative efficiency

The allocative efficiency of a system of auctions is an important market-level property since it characterizes the extent to which the market maximizes social welfare by allocating items to the buyers that value them most. In our setting an analysis of allocative efficiency has particularly important implications since it can help market operators assess the extent to which the, mostly uncoordinated, consumer-to-consumer simultaneous auction markets encountered on the Internet introduce allocative inefficiencies. Allocative inefficiencies are due to imperfections in the matching between bidders and auctions that result from the probabilistic nature of the auction selection equilibrium of Proposition 3.

Specifically, the lack of coordination between bidders makes it possible that two or more high bidder types will cluster on the same auction (in which case only one of them wins and the remaining auctions will be left to lower bidder types) whereas if these same bidders could coordinate and distribute their bids to different auctions they would all win, increasing social welfare. In addition, the probabilistic mating-of-likes equilibrium allows for, say, a low zone bidder type to target and, probabilistically, win a higher reputation ranked seller's auction. The precise notion of efficiency that applies to our setting is given below:

Definition 1: *Consider a setting where* M *sellers with reputations* $r_1 \geq r_2 \geq \ldots \geq r_M$ *simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to* N *unit-demand buyers. Buyers are independently drawn from the same type distribution* F*, place at most one bid and follow a symmetric auction selection strategy* $\mathbf{s}$*. The expected allocative efficiency of the system of auctions under buyer type distribution* F *and strategy* $\mathbf{s}$ *is equal to:*

$$\eta(F,\mathbf{s}) = \frac{\sum_{k=1}^{M} r_k H_{1,k}(F,\mathbf{s})}{\sum_{k=1}^{\min(M,N)} r_k F_{N+1-k,N}}$$

where $H_{1,k}(F,\mathbf{s})$ is the highest bidder's expected type in seller k's auction and $F_{i,N}$ is the expected value of the ith order statistic of a sample of N values independently drawn from distribution F.[3]

The numerator of $\eta(F,\mathbf{s})$ is the expected social welfare resulting from the system of simultaneous auctions. The denominator is the expected maximum social welfare, attainable if, for any randomly drawn set of N buyer valuations, the highest seller is matched with the highest buyer, the second-highest seller is matched with the second-highest buyer, etc.

The following lemma gives a closed form expression for the highest bidder's expected type.

Lemma 2: *Consider a setting where* M *sellers with reputations* $r_1 \geq r_2 \geq \ldots \geq r_M$ *simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to* N *unit-demand buyers, independently drawn from the same type distribution* $F(t)$*. The highest bidder's expected type in seller* k *'s auction is equal to:*

---

[3] We adopt the usual convention that the first order statistic is the minimum of the sample and the Nth order statistic is the maximum.

$$H_{1,k} = 1 - \int_0^1 \left(1 - Q_k(1) + Q_k(t)\right)^N dt \tag{7}$$

There is no closed-form expression for $\eta(F, \mathbf{s})$. However, given M, N, a set of seller reputations and a buyer type distribution F, it can be computed in a straightforward manner from (7) and the formulas of order statistics distributions (David and Nagaraja 2003). We rely on this result to measure the efficiency gap in our experimental market.

3.3 Seller revenue

This section considers the impact of reputation on seller revenue in settings where each seller competes against sellers of different reputation. If the seller has reputation *r*, then his expected revenue from a second price auction is equal to $U = r \times E[\text{secondhighestbidder'stype}]$. We begin by considering the case of a single monopolist seller. It is well known (see, for example, Riley and Samuelson 1981) that the second highest bidder's expected type is a function of the type distribution and equal to:

$$H_2(F) = \int_0^1 [tf(t) + F(t) - 1] F(t)^{N-1} dt \tag{8}$$

Integrating by parts and rearranging, equation (8) can be equivalently rewritten as:

$$H_2(F) = 1 + (N-1)\int_0^1 F(t)^N dt - N \int_0^1 F(t)^{N-1} dt \tag{9}$$

The following Lemma generalizes equation (9) in the case where there are *M* competing sellers offering simultaneous auctions.

Lemma 3: *Consider a setting where* M *sellers with reputations* $r_1 \geq r_2 \geq \ldots \geq r_M$ *simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to* N *unit-demand buyers, independently drawn from the same type distribution* $F(t)$. *The second highest bidder's expected type in seller* k *'s auction is equal to:*

$$H_{2,k}^{\{r_1,\ldots,r_M\}} = 1 + (N-1)\int_0^1 \left(1 - Q_k(1) + Q_k(t)\right)^N dt - N \int_0^1 \left(1 - Q_k(1) + Q_k(t)\right)^{N-1} dt \tag{10}$$

Proposition 4 characterizes the second highest bidder's expected type in a simultaneous auction setting where sellers have different reputations.

Proposition 4: Consider *a setting where a seller with reputation* r_k *(heretoforth referred to as seller k) competes with* $M-1$ *other sellers with reputations* $r_1 \geq \ldots \geq r_{k-1} \geq r_{k+1} \geq \ldots \geq r_M$. *All* M *sellers simultaneously offer sealed-bid, second-price auctions for one unit of a homogeneous good to* N *unit-demand buyers, independently drawn from the same type distribution* $F(t)$. *Let* $H_{2,k}^{\{r_1,\ldots,r_k,\ldots,r_M\}}$ *denote the expected value of the second highest bidder's type on seller k's auction. The following statements are true:*

1. $H_{2,k}^{\{r_1,\ldots,r_k,\ldots,r_M\}}$ *is an increasing function of seller* k*'s own reputation* r.
2. $H_{2,k}^{\{r_1,\ldots,r_k,\ldots,r_M\}}$ *is a decreasing function of every other seller's reputation.*

The main implication of Proposition 4 is that competition amplifies the impact of a reputation on a seller's expected auction revenue $U_k = r_k H_{2,k}^{\{r_1,\ldots,r_M\}}$. When all sellers are identical, a seller's reputation affects the maximum valuation for that seller's good (it is scaled by r_k) but does not affect the second highest bidder's expected type (because all buyers randomize among all sellers). On the other hand, if sellers have different reputations, then sellers with higher (lower) reputation attract more (fewer) bidders of higher valuations and end up with higher (lower) expected second highest bidder types relative to the baseline case of identical sellers. Expected seller revenue is then a convex function of a seller's reputation.

3.4 Multiple bid equilibria

Most auction settings do not limit the number of bids that buyers can simultaneously place on auctions for similar goods. Accordingly this section extends the preceding analysis to a setting where buyers are allowed to bid on an arbitrary number of simultaneous auctions. As before, we restrict the analysis to symmetric Bayes-Nash equilibria, that is, to equilibria where all bidders follow identical strategies. Even though all buyers have unit demand, it is plausible that some may then find it profitable to place several bids. There are two motivations for such behavior. First, by placing multiple bids, buyers increase their chances of winning at least one auction. Second, since sellers are unreliable, by winning more than one auction buyers increase their chances of *receiving* at least one item. On the other hand, if a buyer ends up receiving more than one item, she gets zero utility from additional units but still has to pay the respective auction prices.

When bidders are allowed to place any number of bids, individual bid amounts need no longer be equal to the bidder's respective expected valuations. The problem becomes one of finding a bid vector $\mathbf{b}(t)=(b_1(t),...,b_M(t))\in[0,t]^M$ that maximizes the bidder's expected global surplus from the system of auctions, given her beliefs about everybody else's bidding behavior. Let $G_k(b_k)$ denote a bidder's subjective probability of winning auction k conditional on placing bid b_k on that auction and also conditional on her beliefs about every other bidder's strategy. Let $g_k(b_k)$ denote the corresponding probability density function. A bidder's expected surplus from placing a vector of bids is given by:

$$\begin{aligned}V(t,\mathbf{b}(t))=tPr[&\text{receive item from at least one seller}\,|\,\mathbf{b}(t)]\\&-\sum_{k=1}^{M}E[\text{payment to seller } k\,|\,b_k(t)]\end{aligned}\tag{11}$$

or, more formally, by:

$$V(t,\mathbf{b}(t))=t\left[1-\prod_{k=1}^{M}(1-r_kG_k(b_k(t)))\right]-\sum_{k=1}^{M}\int_0^{b_k(t)}xg_k(x)dx\tag{12}$$

In the above expression $1-r_kG_k(b_k(t))$ is the probability of *not receiving* an item from seller k. This includes the probability of not winning seller k's auction plus the probability of winning the auction but not receiving the item because seller k cheats. Accordingly, $\prod_{k=1}^{M}(1-r_kG_k(b_k(t)))$ is the probability of not receiving the item from any seller and $1-\prod_{k=1}^{M}(1-r_kG_k(b_k(t)))$ the probability of receiving the item from at least one seller. The expression $\int_0^{b_k(t)}xg_k(x)dx$ is the well-known expression for the expected payment of a single second-price auction when bidding $b_k(t)$. Therefore,

$\sum_{k=1}^{M}\int_{0}^{b_k(t)} x g_k(x)dx$ represents the buyer's total expected costs of participating to the system of auctions.

The following result shows that maximization of (12) implies non-zero bids in all available auctions for all bidder types in the interior of the type distribution:

Proposition 5: *For all* $t \in (0,1)$*, any bid vector* $\mathbf{b}(t)$ *that maximizes (12) must have* $b_k(t) > 0$ *for all* $k = 1,\dots,M$ *.*

The form of the surplus-maximizing equilibrium bid vector can be obtained by setting the partial derivatives of (12) to zero:

$$\frac{\partial V(t,\cdot)}{\partial b_k} = g_k(b_k(t))\left[tr_k \prod_{j\in\{1,\dots,M\}-\{k\}} (1 - r_j G_j(b_j(t))) - b_k(t) \right] = 0 \tag{13}$$

If type t has positive probability density then at any symmetric equilibrium it must be $g_k(b_k(t)) > 0$. To see this, recall that $g_k(b_k(t))$ is equal to the probability that *some* bidder will post a bid equal to $b_k(t)$ on auction k. If $b_k(t)$ is part of one of type t's equilibrium bid vectors (there might be multiple such vectors in the case of mixed strategies) then it must be chosen by type t with positive probability, which implies that $g_k(b_k(t)) > 0$. The second part of (13) then yields:

$$b_k(t) = (tr_k) \prod_{j\in\{1,\dots,M\}-\{k\}} (1 - r_j G_j(b_j(t))) \tag{14}$$

In words, type t's optimal bid in auction k is equal to the bidder's expected valuation tr_k multiplied by the probability *of not receiving the item through any of the other auctions*, given the bidder's other bids $b_j(t)$ and her correct probability assessment $G_j(b_j(t))$ of winning each auction if every other bidder follows the same bidding strategy. The reader can verify that this expression is also equal to the bidder's marginal utility of winning auction k, given bids $b_j(t)$ in all other auctions. It is easy to show that the second partial derivative of (12) is negative, confirming that (14) is indeed the solution that maximizes (12).

The conceptual simplicity of recursive equation (14) is deceiving, since $G_j(b)$ depends on every other bidder's strategy over the entire type space, which (as it turns out) might be mixed and non-monotone. Observe that, for a given set of seller reputations r_k, each bid $b_k(t)$ can be equivalently characterized by its bid-to-valuation ratio:

$$\beta_k(t) = \frac{b_k(t)}{tr_k} = \prod_{j \in \{1,\dots,M\}-\{k\}} (1 - r_j G_j(b_j(t))) \tag{15}$$

This alternative characterization has the advantage of mapping the range of every auction's possible bids into the unit interval. From (15) it follows that a buyer's bid-to-valuation on each auction is negatively correlated with every one of her other bids. Therefore, the more a bidder focuses on winning a particular auction, the lower the bids she places on all other auctions.

In the most general case, bidding strategies will be mixed and, therefore, defined by a probability density function $\sigma_k(t,\beta):[0,1]\times[0,1]^M \to [0,1]$, where $\beta = (\beta_1,\dots,\beta_M)$. Mixed strategy profiles that satisfy (15) can be computed using Monte Carlo methods.[4] While the details of multiple bid equilibria are substantially more complex, in spirit they turn out to possess similar properties to the mating-of-likes one-bid equilibrium detailed earlier in this section. The lowest bidder types place bids close to their expected valuation on all auctions, since they have very low probability of winning any auction. Equilibrium bid vectors of higher bidder types have one of two forms:

Type I) a "serious" or high bid (i.e. a bid whose bid-to-valuation ratio is close to one) on one auction and low bids on all remaining auctions or

Type II) intermediate bids (i.e bids whose bid-to-valuation ratios take intermediate values) on all auctions.

Bidders randomize between using Type I and Type II bid vectors. When using Type I vectors they further randomize with respect to which auction they place their high bid on. As their type increases, bidders use Type I bid vectors more often and place their high bid on higher reputation auctions with higher probability. Just as in the one-bid case, the multi-bid equilibrium, thus, implements a form of probabilistic positive assortative matching between buyers and sellers.

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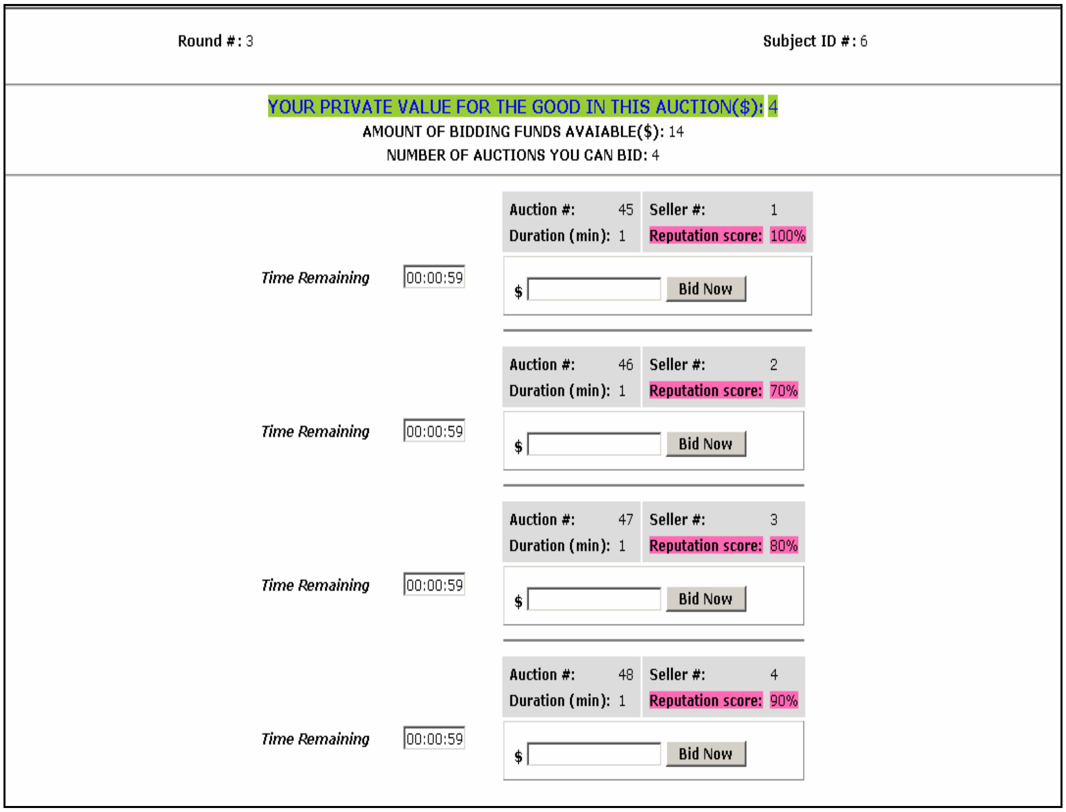
[4] Due to space limitations the results of our numerical simulations of multi-bid equilibria are omitted from this manuscript but are reported on the following working paper: Dellarocas, C. Simultaneous Auctions of Imperfect Substitute Goods by Sellers of Different Reputations(March 7, 2008). Robert H. Smith School Research Paper No. RHS 06-057. Available at SSRN: http://ssrn.com/abstract=1115805

4. Laboratory Experiment

An effective way to compare actual behavior with theoretical predictions is by conducting controlled laboratory experiments. Our laboratory experiment uses economic incentives drawing from the work of Vernon Smith (1976) on induced value theory. This allows us to control for heterogeneous values and study the decision making behavior of bidders in a controlled setting. Experimental work on auctions has provided valuable observations of bidding behavior that would not otherwise be possible in a less controlled environment (Cox et al. 1982; Kagel and Levin 1985; Kagel et al. 1987; Kagel and Levin 2002; Kagel, Ham and Lehrer 2005; Kagel and Levin 2005).

4.1 Experimental Setting

Our experimental setting emulates a market where six unit-demand bidders face four sellers of different reputation scores (100%, 90%, 80% and 70%), each selling one unit of an identical good. Our subjects are MBA students at a top 20 global business school. All subjects assume the role of a bidder; the seller's role is computerized. Consistent with Bolton et al. (2004), the seller's reputation score indicates the probability that the buyer will receive the purchased good as advertised. This means that if a bidder wins an auction from a seller with a 90% reputation score the probability that the winner will receive their good is 0.9; however, regardless of whether or not the good "ships" the winner must always pay for the unit won. In each round bidders may bid on as many seller auctions as they choose. However, the assumption of unit demand implies that if a bidder wins two units she must pay for both even though the second unit is of no value to her. Once bidders have submitted bids a subsequent screen indicates whether they have won the good and shows their profit in that round. The experiment lasts for twenty rounds, and bidders are randomly assigned different private values in each round, drawn from a uniform distribution with support U[6,10]. Figure 2 depicts the experimental setup.



Round #: 3

Subject ID #: 6

YOUR PRIVATE VALUE FOR THE GOOD IN THIS AUCTION($): 4

AMOUNT OF BIDDING FUNDS AVAIABLE($): 14

NUMBER OF AUCTIONS YOU CAN BID: 4

Auction #: 45
Seller #: 1
Duration (min): 1
Reputation score: 100%
Time Remaining 00:00:59
$

Auction #: 46
Seller #: 2
Duration (min): 1
Reputation score: 70%
Time Remaining 00:00:59
$

Auction #: 47
Seller #: 3
Duration (min): 1
Reputation score: 80%
Time Remaining 00:00:59
$

Auction #: 48
Seller #: 4
Duration (min): 1
Reputation score: 90%
Time Remaining 00:00:59
$

Figure 2: Screenshot of Experimental Setup

Subjects are paid in cash at the end of the experiment. Total profits for each bidder *i*, denoted by $Profit_i$ are calculated as follows:

$$Profit_i = \sum_{t=1}^{20}\left(Value\_Received_{it} - Price\_Paid_{it}\right) + Participation\_Fee$$

In the above expression:

1. $Value\_Received_{it}$ is equal to bidder *i*'s private value at round *t* if *at least one* good is received by that bidder at round *t* or zero otherwise.

2. $Price\_Paid_{it}$ is the sum of prices paid at all auctions won by bidder *i* at round *t*

3. $Participation\_Fee$ is the fixed amount subjects are paid for participating in the auction.

We ran two identical repetitions (labeled A and B in the subsequent analysis), each with a distinct group of six bidders and four computerized sellers. Each session lasted for twenty rounds. Prior to actual play-for-pay, bidders were given detailed instructions on the experiment's objective and asked to answer a quiz (details of the quiz and instructions are available to future researchers from the authors on request) on how to calculate payoffs under the different scenarios of winning (not winning) and receiving (not receiving) a single unit from participating in multiple auctions. The experimenter individually ensured that all subjects understood all questions in the quiz and gave personalized explanations to all those that did not. Three training rounds were used to make subjects familiar with the web-based auction environment, the associated risk and the payoff realization. We also polled subjects on their familiarity with Internet markets and found a high degree of awareness and prior participation in online auctions and e-commerce activities. Seventy five percent of our subjects had either bought or sold something on-line in the last 30 days and the average amount spent per month on-line purchases was between \$50 and \$100.

4.2 Analysis of Results

The analysis of Section 3 provides crisp predictions on bidding strategies, impact of reputation on seller revenue and allocative efficiency of the system of simultaneous auctions. This section discusses how our experimental results compare to theory in each of these three areas.

Bidding Strategies: Our one-bid analysis suggests that bidder types will be positively correlated with seller reputation, with higher types more likely to bid on higher reputation sellers. In the absence of restrictions regarding the number of bids our theory further predicts that, if there are no auction participation costs (including cognitive costs), all bidders will place non-zero (but potentially very low) bids on all auctions but higher bidders will be more likely to place a "serious" bid (i.e. a bid whose bid-to-valuation ratio is close to one) on a high reputation seller.

To compare our experimental findings with the above theoretical predictions we broke down bidding strategies by number of simultaneous bids and calculated the average bidder's private value (normalized from 0 to 100 for expositional ease), average bid-to-valuation ratio and average round where each bidding strategy was observed. We did this separately for each of the two repetitions – just in case there are any repetition-specific anomalies. Table 1 presents the results:

Repetition	# of Simultaneous Bids	# of Cases where Observed	Average Round where Observed	Average of Bidder's Value	Average Bid-to-Valuation Ratio
A	0	9	12.78	18.92	0.00
A	1	91	10.54	53.36	1.01
A	2	20	9.30	38.66	0.91
B	1	94	10.72	49.53	1.13
B	2	25	10.04	47.70	0.97
B	3	1	1.00	76.75	0.88

Table 1: Majority of Bidders Place a Single Bid

In both sessions bidders place exactly one bid about 75%-78% of the time and two bids about 20% of the time. There is only one case of 3 bids (in repetition B) and 9 cases of 0 bids (all in repetition A and by the same low valuation bidders). The case of zero bids is irrational and we interpret it as an anomaly that does not merit further discussion. We observe that, even though there was no restriction on the number of bids, most bidders choose to place only one bid. We attribute this behavior to the presence of cognitive bidding costs (Easley and Tenorio 2004) associated with calculating a bid vector that satisfies equation (15). From the Proof of Proposition 5, each additional bid provides some amount of marginal utility but also adds one more equation to the system (15); this increases the cognitive cost of calculating the vector of bids. The point at which a bidder's marginal cognitive cost exceeds the corresponding marginal utility determines the optimal number of bids for that bidder.

Interestingly, the average bidder type (valuation) in the case of two bids is lower than the average bidder type in the case of one bid, suggesting that low valuation bidders are more likely to place multiple bids. Such behavior is consistent with the above cognitive cost explanation. Recall, from Section 3.4, that the marginal utility of placing an additional bid is proportional to the probability of not winning and receiving the good from any other auction, given the bidder's other bids. *Ceteris paribus* this probability is likely to be higher for low value bidders, whose other bids are less likely to be successful. The marginal utility of placing additional bids is, thus, higher for low value bidders.

It is also interesting to note that the bid-to-valuation ratio is higher in the case of one bid than it is in the case of two bids, suggesting, again consistent with theory, that when people place multiple bids they scale at least one of them down relative to their expected valuation: very often the second bid is a "lowball" bid. In the case of one bid the average bid-to-valuation is above one, suggesting that some people overbid. There is no apparent pattern that relates to average round.

We further broke down bidding strategies by examining the "highest reputation seller" where bidders place a nonzero bid. The results are summarized in Table 2:

Repetition	# of Simultaneous Bids	# of Cases where Observed	Highest Seller Reputation	Average Round where Observed	Average of Bidder's Value	Average Bid-to-Valuation Ratio
A	0	9	0.00	12.78	18.92	0.00
A	1	56	1.00	9.82	58.90	1.00
A	1	15	0.90	11.53	47.50	1.07
A	1	9	0.80	15.11	47.44	0.92
A	1	11	0.70	9.09	37.98	1.10
A	2	12	1.00	8.50	45.13	0.90
A	2	8	0.90	10.50	28.97	0.92
B	1	31	1.00	11.23	64.16	1.01
B	1	20	0.90	11.30	49.08	1.15
B	1	25	0.80	8.76	45.29	1.14
B	1	18	0.70	11.94	30.71	1.29
B	2	20	1.00	9.55	40.96	0.98
B	2	3	0.90	11.67	88.58	0.93
B	2	2	0.80	12.50	53.75	0.92
B	3	1	1.00	1.00	76.75	0.88

Table 2: Bidders Conform to Probabilistic Assortative Matching Equilibria

Three observations, valid across both repetitions, are particularly noteworthy: First, when bidders place one bid only (# of simultaneous bids = 1), the average bidder private value monotonically declines with the seller's reputation. That is, as theory expects, higher bidders are more likely to place a bid on a higher seller's auction, and lower bidder types are more likely to target lower reputation sellers. Second, in the single bid case, the average bid-to-valuation ratio appears to increase for lower reputation sellers. This suggests that when bidders choose to bid on lower reputation sellers they fail to adjust for the associated risk. Third, bids to the next-to-last seller (0.8) have a distinctly lower bid-to-valuation ratio compared to bids to the lowest seller (0.7). This indicates a potentially interesting "focal" anomaly whereby bidders place disproportionate focus on the lowest seller and "bypass" the 3rd seller.

Figure 3 shows two panels that provide the reader with a graphical depiction of the probabilistic assortative matching pattern. The left panel depicts the (normalized) average bidder type as a function of the reputation of the seller with whom the bidders chose to bid on. The chart shows that on average, sellers with higher reputations attracted higher type bidders. The right panel contains the empirical distributions, as displayed by the boxplots, of the sellers chosen by the buyers falling in the various zones[5] of the probabilistic assortative matching. The reader is encouraged to relate this to Figure 1

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[5] Recall from Proposition 3 that zone delimiters t_z, $z = 0,1,..,L-1$ are solutions of the following equation:

wherein we predicted, and herein we observe, that zone 1 bidders restrict their attention to the top-most reputation sellers. This is evident in the horizontal line at Seller Reputation 1 corresponding to Zone 1. As predicted, there is no spread in the distribution of the sellers targeted by the highest zone type buyers. Zone 2 bidders are expected to randomize between the top two reputation sellers, but it appears that they randomize (as is evident by the range or spread depicted in the box-plots' whiskers) between the top three sellers, ignoring the lowest reputation seller. Zone 3 bidders are also observed to randomize between the top three sellers, with the exception of a few outliers denoted by the "***" at the lowest reputation sellers. Consistent with theoretical predictions Zone 4 (the lowest type) bidders are observed to randomize between all four sellers but with higher probability mass associated with sellers of lower reputations. We believe this to be strong evidence of the randomization that is central to the probabilistic mating-of-like equilibrium we proposed.

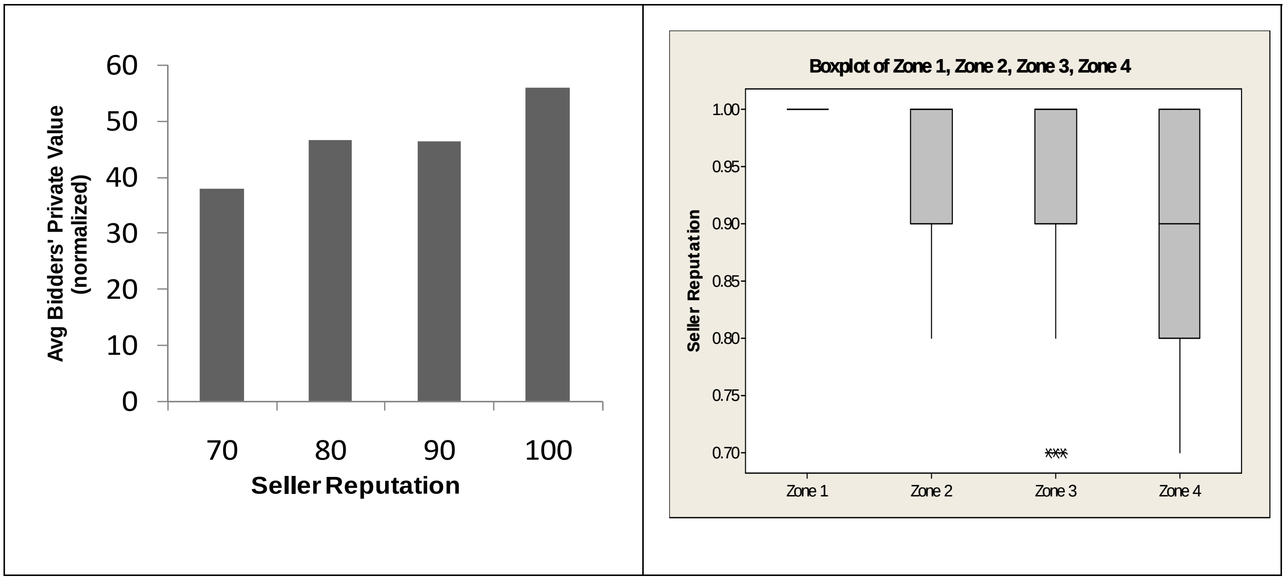

Figure 3: Graphical Evidence of Probabalistic Assortative Matching

We also examined how bid-to-valuation ratios depended both on the seller's reputation as well as on the bidder's private value. Figure 4 plots bid-to-valuation ratios (in the cases where bidders place one bid only) for each of the 4 sellers as a function of the bidder's private value.

$$F(t_z) = \left(\sqrt[N-1]{r_{z+1}} \sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}} \right) - (z-1)$$

Figure 4: Low Bidder Types Fail to Appropriately Adjust for Risk Associated with Risky Sellers

The bid-to-valuation ratio for seller 1.0 is very close to 1 for all bidders, i.e. all bidders bid very close to their expected valuation. However, as we move towards lower-reputation sellers the bid-to-valuation appears to increase as we move down the bidder's valuation, i.e. low bidders appear to overbid on low sellers. We postulate that this effect is a result of competition. Low value bidders want to increase their chances of winning *some* auction and therefore, overbid, especially on low sellers. However, this is also evidence that they fail to adjust for the risk associated with lower reputation sellers. We explore this further in after discussing seller revenue.

Seller Revenue: Our analysis predicts that, in simultaneous auction settings where sellers have different reputations, seller revenue will be an increasing convex function of reputation. Table 3 displays the average seller revenue per seller and repetition. Using equation (10) we also calculated the revenue that theory would predict for each seller in our experimental setting.

Seller Type (Reputation)	Observed Average Seller Revenue (per round)			Observed Risk-Adjusted Average Revenue*	Theoretically Predicted Risk-Adjusted Revenue*
	Repetition A	Repetition B	Average across both Repetitions		
100	8.34	7.75	8.04	8.04	6.11
90	4.43	5.07	4.75	5.27	5.85
80	3.01	4.63	3.82	4.77	5.55
70	4.07	4.95	4.51	6.44	5.20

* Risk-adjusted Average Revenue = Average Revenue / Seller's Reputation

Table 3: Seller Revenue as a Function of Reputation: Theory vs. Experiment

Two results stand out. First, the top seller makes substantially more than the other three sellers, even if we adjust for seller reputation, and certainly more than what theory would predict. Our preceding analysis indicated that bid amounts on the top seller are as expected, that is, equal to each bidder's private value. Nevertheless, bidders choose to bid at the top seller more often than "they should" in an efficient equilibrium and this results in higher seller revenue. Seeking safety, bidders appear to not properly account for the increased competition at the top. Second, the lowest-reputation seller (70%) makes more than the next to last (80%), contradicting theoretical predictions. This appears to be consistent across our two repetitions and suggests the presence of a behavioral anomaly. We hypothesize that bidders who want to hedge their bets by bidding on a "low" seller focus too much on the bottom seller and bypass the next-to-last option.

Prior work can help provide tentative explanations for the unexpected flocking of bidders to the high and low reputation sellers. Our auction setting contains an element of risk, as bids placed on any of the lower reputation sellers pose the risk that the purchased good will not ship. Studies on decision making under risk show that expected utility theory can be limited in explaining decision making behavior in this setting (Weber and Camerer 1987; Fishburn 1988). For example, is often the case that when individuals chose among risky alternatives the weight they assign to each outcome may not correspond to the actual probability of that outcome (Wu and Gonzalez 1996). Work in the decision sciences posits that when making decisions under risk or uncertainty individuals will tend to disproportionately weight the extreme tails of the outcome distribution, explaining the disconnect between what is observed and what is supported by expected utility theory. It could be the case that as a result of probability weighting biases, subjects bidding on the low reputation sellers could possibly have over-estimated the probability of actually receiving the good they purchased, particularly when comparing this to the certain payoff of zero if no bid is placed. The result is that some bidders may have placed a higher probability on the likelihood of winning and receiving the good from the lowest reputation seller; their bids reflected this belief.

Second, overweighting the probability of losses when anchoring on the high reputation seller could explain why we observed a greater than expected number of bids placed in the 100% reputation sellers' auctions. If bidders perceived the high reputation sellers as representing a certain successful completion of the transaction, these bidders would tend to over react to small decreases in the probability of receiving the good from a lower reputation seller. The result is that these bidders would be more likely to ignore lower reputation sellers almost completely and bid only on the 100% seller.

Table 4 explores these two findings further by comparing the expected proportion[6] of bids placed on a seller with a given reputation with the observed. It is evident that a higher than theoretically expected proportion of bids were placed on the highest reputation sellers, and that bidders tend to bid above their expected value on the lowest reputation seller.

Seller Reputation	**100%**	**90%**	**80%**	**70%**
Pct of Bids per Round (Theory)	28%	26%	24%	22%
Pct of Bids per Round (Experiment)	43%	22%	15%	20%
Bid-to-valuation Ratio (Experiment)	0.99	0.97	0.99	1.08

Table 4: Theoretical v Observed Bids and Bid-to-Valuation Distribution

Crowding of bidders on the top seller and overbidding on the lowest seller result in bidder surplus that is roughly 60%-70% of what theory would predict. The encouraging news is that this distortion tends to ameliorate over time as bidders gain experience with the setting and tend to focus less on the extreme sellers in later rounds of the auction.

---

[6] This is an approximation based on the one-bid setting (which appears to be played three quarters of the time) for which closed form solutions exist.

Allocative Efficiency: As we discussed, simultaneous auctions are inherently inefficient because, in the absence of complete information about each other's private value, bidders cannot coordinate with each other when choosing where to bid. It is then possible that two or more high value bidders who could all win if they could coordinate and choose adjacent auctions will cluster on the same auction, reducing efficiency. For instance, the theoretical allocative efficiency of our four-seller, six-bidder experimental setting (calculated using the formulae introduced in Section 3.2) is 80%.

Figure 5 shows that the experimentally observed efficiency levels in the 20 rounds of our experiments (averaged over both repetitions) are remarkably close to those predicted by theory.

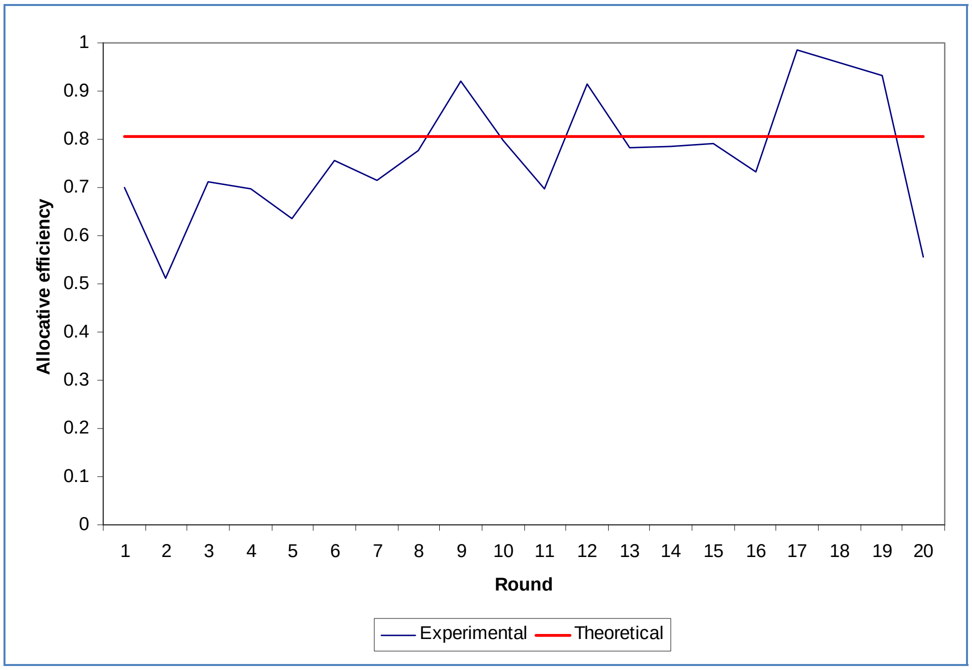

Figure 5: Theoretical vs. experimentally observed allocative efficiency.

Overall, we find significant conformance to our theoretical predictions as well as some interesting anomalies. Most importantly we find that vertically-differentiated simultaneous auctions are subject to significant coordination failures, inherent in their design. These failures are exacerbated by crowding on the top seller and overbidding on the bottom seller. We believe more can be done by auctioneers to explicitly recognize and reduce the cognitive challenges faced by bidders in simultaneous auctions with varying reputation sellers. We discuss these managerial implications as well as directions for future research in the next section.

5. Conclusions and Directions for Future Research

This paper derives the form of bidding equilibria in vertically-differentiated simultaneous sealed-bid, second price auction settings where sellers offer single units of identical goods, buyers have unit demand, and sellers have different reputations for upholding their end of the transaction. Among other possible applications, the model can serve as an abstraction of large-scale decentralized Internet auction marketplaces, such as eBay.

If bidders are restricted to place at most one bid, the unique Bayes-Nash equilibrium corresponds to a form of probabilistic positive assortative matching: Bidders rank-order auctions with respect to their relative attractiveness, taking into account their type and the seller's reputation. Bidders assess where they stand on the valuation type space relative to other bidders and assign higher probability to bidding on the auction that "matches" their respective type zone, while also occasionally bidding on "higher" auctions. If bidders are not restricted to place one bid, bidding equilibria are substantially more complex but generally maintain a probabilistic mating-of-likes (Becker 1973) pattern in that higher value bidders are more likely to place a "serious" (i.e. high bid-to-valuation ratio) bid on a high seller. Our experimental data is generally consistent with such behavior.

The probabilistic nature of the equilibrium introduces allocative inefficiencies. Specifically, the lack of coordination between bidders makes it possible that two or more high bidder types will cluster on the same auction (in which case only one of them wins and the remaining auctions will be left to lower bidder types) whereas if these same bidders could coordinate and distribute their bids to different auctions they would all win, increasing social welfare. Our experimental data shows that these coordination failures are exacerbated by bidders' tendency to crowd on the top (highest reputation) seller and to overbid on the bottom (lowest reputation) seller. Although crowding and overbidding improved in later rounds, cumulatively these phenomena resulted in bidder surplus that was 60%-70% of what theory would predict.

The above inefficiencies constitute the "price of anarchy" of uncoordinated auction markets. In theory, it is possible to solve this problem completely by centralizing all the simultaneously occurring single-item auctions into a single multi-item VCG auction and asking bidders to submit menus of bids for any subset of the available items. In settings where the number of bidders and the number of auctions are roughly equal (eBay attracts approximately three bidders per auction (Bapna et al 2008)) this might be a sensible suggestion for auction marketplace operators to consider. We expect the recent advances in the study of combinatorial auctions (Jones et al 2006) to further help model a centralized version of this problem.

We expect our study to fuel future work in three areas. Given the growing importance of simultaneous auctions in settings with uncertainty, we expect future work to further study the behavior of individuals in such markets. Given some recent advances in researchers learning about bidder's types through revealed preferences on sniping agents (Bapna et al 2008), we are working on an empirical design that parallels our experimental setup. Future work will also benefit from closely examining empirical bidding strategies in settings similar to ours. Together, the current study and related future work will help inform practitioners on how to improve the design of mechanisms similar to the ones considered here. We expect future work to examine whether the general approach of probabilistic assortative matching brought forth in this paper carries over to other emerging electronic matching "markets" such as those for dating and marriage, or the increasingly important market for sponsored search keywords (Edelman et al. 2007). Reminiscent of the now famous "bar scene" in the movie "A Beautiful Mind" we see parallels in that there are suitors of varying types simultaneously competing for matches with varying attractiveness. While similar coordination failures can be expected in such settings, we argue that marginal improvements in efficiency of such markets will offer significant and disproportionate improvement on peoples' lives and happiness quotients.

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Appendix A - Proofs[7]

1. Proofs of Lemmas

Lemma 1

Part 1.

Follows directly from the interpretation of the quantity $Q_k(1|\mathbf{s})$.

Part 2.

The probability that a buyer of type *t* wins the auction of seller *k* is equal to:

$$W_k(t|\mathbf{s}) = \sum_{m=0}^{N-1} Pr[m \text{ other bidders choose } k|\mathbf{s}] \times Pr[\text{all } m \text{ other bidders have types} \leq t|\mathbf{s}] \quad (19)$$

From Part 1 it follows that $Pr[m\text{otherbidderschoose}k|\mathbf{s}] = \binom{N}{m} Q_k(1|\mathbf{s})^m \left(1 - Q_k(1|\mathbf{s})\right)^{N-m}$. The subjective probability that a bidder who selects seller *k* has type *t* is given by:

$$g_k(t|\mathbf{s}) = Pr[t|k;\mathbf{s}] = Pr[k|t;\mathbf{s}] f(t)/Pr[k|\mathbf{s}] = s_k(t) f(t) / \int_0^1 s_k(t) f(t) dt = Q'_k(t|\mathbf{s})/Q_k(1|\mathbf{s})$$

The probability that all other *m* bidders of auction *k* have types less than or equal to *t* is equal to:

$$Pr[\text{all}m\text{otherbiddershavetypes} \leq t] = \left(\int_0^t g_k(u|\mathbf{s}) du\right)^m = \frac{Q_k(t|\mathbf{s})^m}{Q_k(1|\mathbf{s})^m}$$

Substituting into (19) and making use of the properties of binomial sums:

$$W_k(t|\mathbf{s}) = \sum_{m=0}^{N-1} \binom{N}{m} Q_k(1|\mathbf{s})^m \left(1 - Q_k(1|\mathbf{s})\right)^{N-m} \frac{Q_k(t|\mathbf{s})^m}{Q_k(1|\mathbf{s})^m} = \left(1 - Q_k(1|\mathbf{s}) + Q_k(t|\mathbf{s})\right)^{N-1}$$

Part 3.

---

[7] For the benefit of the reviewers, we present the complete proofs. If space is an issue we can condense this section by providing proof sketches and delegate the full proofs to an online appendix.

Buyer *t*'s expected surplus from selecting seller *k* is equal to:

$$\begin{aligned} V_k(t \mid \mathbf{s}) = \sum_{m=0}^{N-1} & Pr[m \text{ other bidders choose } k \mid \mathbf{s}] \times Pr[\text{all } m \text{ other bidders have types} \le t \mid \mathbf{s}] \\ & \times r_k(t - E[\text{2nd highest type} \mid \text{highest type} = t; \exists m \text{ other bidders}, \mathbf{s}]) \end{aligned} \qquad (20)$$

The conditional probability that the 2nd highest out of $m+1$ bidder types is equal to x if the highest type is equal to t is equal to:

$$Pr[\text{2ndtype} = \text{x} \mid \text{1sttype} = t; m+\text{1bidders}, \mathbf{s}] = \frac{Pr[\text{2ndtype} = \text{xand1sttype} = t; m+\text{1bidders}, \mathbf{s}]}{Pr[\text{1sttype} = t; m+\text{1bidders}, \mathbf{s}]}$$

where, from the theory of order statistics (see, for example, David and Nagaraja 2003):

$$\begin{gathered} Pr[\text{1sttype} = t; m+\text{1bidders}, \mathbf{s}] = G_k(t \mid \mathbf{s})^m G'_k(t \mid \mathbf{s})(m+1) \\ Pr[\text{2ndtype} = \text{xand1sttype} = t; m+\text{1bidders}, \mathbf{s}] = G_k(x \mid \mathbf{s})^{m-1} G'_k(x \mid \mathbf{s}) G'_k(t \mid \mathbf{s})(m+1)m \end{gathered}$$

Therefore:

$$\begin{gathered} Pr[\text{2ndtype} = \text{x} \mid \text{1sttype} = t; m+\text{1bidders}, \mathbf{s}] \\ = \frac{G_k(x \mid \mathbf{s})^{m-1} G'_k(x \mid \mathbf{s}) G'_k(t \mid \mathbf{s})(m+1)m}{G_k(t \mid \mathbf{s})^m G'_k(t \mid \mathbf{s})(m+1)} = \frac{G_k(x \mid \mathbf{s})^{m-1} G'_k(x \mid \mathbf{s})m}{G_k(t \mid \mathbf{s})^m} = \frac{1}{G_k(t \mid \mathbf{s})^m} \frac{partial G_k(x \mid \mathbf{s})^m}{\partial x} \end{gathered}$$

and:

$$\begin{gathered} E[\text{2ndtype} = \text{x} \mid \text{1sttype} = t; m+\text{1bidders}, \mathbf{s}] \\ = \frac{\int_0^t x \frac{\partial G_k(x \mid \mathbf{s})^m}{\partial x} dx}{G_k(t \mid \mathbf{s})^m} = \frac{t G_k(t \mid \mathbf{s})^m - \int_0^t G_k(x \mid \mathbf{s})^m dx}{G_k(t \mid \mathbf{s})^m} = t - \frac{\int_0^t G_k(x \mid \mathbf{s})^m dx}{G_k(t \mid \mathbf{s})^m} \end{gathered}$$

Substituting into (20) we obtain:

$$
\begin{aligned}
V_k(t\,|\,\mathbf{s}) &= \sum_{m=0}^{N-1} \binom{N-1}{m} Q_k(1\,|\,\mathbf{s})^m \left(1-Q_k(1\,|\,\mathbf{s})\right)^{N-1-m} G_k(t\,|\,\mathbf{s})^m\, r_k\, \frac{\int_0^t G_k(x\,|\,\mathbf{s})^m\, dx}{G_k(t\,|\,\mathbf{s})^m} \\
&= r_k \sum_{m=0}^{N-1} \binom{N-1}{m} \left(1-Q_k(1)\right)^{N-1-m} \int_0^t Q_k(x\,|\,\mathbf{s})^m\, dx \\
&= r_k \int_0^t \left(1-Q_k(1\,|\,\mathbf{s})+Q_k(x\,|\,\mathbf{s})\right)^{N-1} dx
\end{aligned}
$$

Lemma 2

From standard auction theory the highest bidder's expected type in a single auction setting with m bidders independently drawn from distribution $F(t)$ is $\int_0^1 t(m-1)\, f(t) F(t)^{m-1}\, dt = 1 - \int_0^1 F(t)^m\, dt$. The highest bidder's expected type in a setting with *M* simultaneous auctions and *N* sellers is equal to:

$$
H_{1,k} = \sum_{m=0}^{N} Pr[m \text{ bidders choose } k] \times E[\text{highest bidder} \,|\, m \text{ bidders}]
$$

where:

$$
Pr[m \text{ other bidders choose } k] = \binom{N}{m} Q_k(1)^m \left(1-Q_k(1)\right)^{N-m}
$$

$$
E[\text{highest bidder} \,|\, m \text{ bidders}] = 1 - \int_0^1 G_k(t)^m\, dt,\; G_k(t) = Q_k(t)\,/\,Q_k(1)
$$

Substituting and rearranging, taking into account the properties of binomial sums, produces the result.

Lemma 3

The second highest bidder's expected type in a setting with *M* simultaneous auctions and *N* sellers is equal to:

$$
H_{2,k} = \sum_{m=0}^{N} Pr[m \text{ bidders choose } k] \times E[\text{second highest bidder} \,|\, m \text{ bidders}]
$$

where:

$$Pr[m \text{ other bidders choose } k] = \binom{N}{m} Q_k(1)^m \left(1 - Q_k(1)\right)^{N-m}$$

$$E[\text{second highest bidder} \mid m \text{ bidders}] = 1 + (m-1)\int_0^1 G_k(t)^m \, dt - m\int_0^1 G_k(t)^{m-1} dt, \; G_k(t) = Q_k(t) / Q_k(1)$$

Substituting and rearranging, taking into account the properties of binomial sums, produces the result.

2. Proofs of Propositions

Proposition 1

The proof is by contradiction. Suppose that there exists a pure strategy one-bid equilibrium in which types $t \in (0, t_0]$ always bid on seller *k*'s auction. There are two possible cases:

1. $t_0 = 1$.

Then $Q_k(t) = F(t)$, $Q_k(1) = 1$ and $Q_\ell(t) = 0$ for all $\ell \neq k$ and all $t \in (0,1]$. From incentive compatibility constraint (6), since it is $V_k(0) = V_\ell(0) = 0$, the assumption $s_k(0) = 1$ implies that $\partial V_k(0)/\partial t \geq \partial V_\ell(0)/\partial t$ for all $\ell \neq k$. Substituting the above values of $Q_k(t)$ into (5) the inequality becomes:

$$\begin{gathered} r_k\left(1 - Q_k(1) + Q_k(0)\right)^{N-1} \geq r_\ell\left(1 - Q_\ell(1) + Q_\ell(0)\right)^{N-1} \\ r_k\left(1 - 1 + 0\right)^{N-1} \geq r_\ell\left(1 - 0 + 0\right)^{N-1} \\ 0 \geq r_\ell \end{gathered}$$

The above is a contradiction if there exists at least one other seller with nonzero reputation.

2. $t_0 < 1$.

As before, the assumption $s_k(t) = 1$ for $t \in (0, t_0]$ requires that $\partial V_k(0)/\partial t \geq \partial V_\ell(0)/\partial t$. This is equivalent to:

$$r_k\left(1 - Q_k(1)\right)^{N-1} \geq r_\ell\left(1 - Q_\ell(1)\right)^{N-1} \tag{21}$$

The assumption of a pure equilibrium implies that, at type t_0, buyers will switch from seller *k* to another seller $\ell \neq k$. If strategies are piecewise continuous then expected buyer surpluses are continuous. The switch, therefore, implies that $V_k(t_0) = V_\ell(t_0)$ and $\partial V_k(t_0)/\partial t \leq \partial V_\ell(t_0)/\partial t$. The assumption $s_k(t) = 1$ for $t \in (0, t_0]$ implies $Q_k(t) = F(t)$ and $Q_\ell(t) = 0$ for all $t \in (0, t_0]$. The inequality of derivatives at $t = t_0$ is, then, equivalent to:

$$r_k\left(1 - Q_k(1) + F(t_0)\right)^{N-1} \leq r_\ell\left(1 - Q_\ell(1)\right)^{N-1} \tag{22}$$

Noting that $r_k\left(1 - Q_k(1) + F(t_0)\right)^{N-1} > r_k\left(1 - Q_k(1)\right)^{N-1}$, it is easy to see that requirements (21) and (22) lead to a contradiction.

Proposition 2

Part 1.

Let $0 < t_0 < t_1 \leq 1$. From Proposition 1 we have established that all equilibria must be mixed at $t \in (0, t_0]$ for some positive t_0. Furthermore we have shown that sellers that are not chosen with positive probability by the lowest buyer types will not be chosen with positive probability by any buyer. I therefore restrict my attention on the subset S of sellers that are included in the choice set of the lowest-valuation bidders. Assume the existence of an equilibrium where $s_k^*(t) = 1$ for $t \in (t_0, t_1]$. For all $\ell \in S$, at $t = t_0$ it must be $V_k(t_0) = V_\ell(t_0)$ (equilibrium is mixed for all lower types) and $\partial V_k(t_0)/\partial t \geq \partial V_\ell(t_0)/\partial t$, or equivalently:

$$r_k\left(1 - Q_k(1) + Q_k(t_0)\right)^{N-1} \geq r_\ell\left(1 - Q_\ell(1) + Q_\ell(t_0)\right)^{N-1}$$

The assumption $s_k^*(t) = 1$ for $t \in (t_0, t_1]$ implies $Q_k(t) = Q_k(t_0) + F(t) - F(t_0) > Q_k(t_0)$ and $Q_\ell(t) = Q_\ell(t_0)$ for all $t \in (t_0, t_1]$. But this, in turn, implies that $\partial V_k(t)/\partial t > \partial V_k(t_0)/\partial t \geq \partial V_\ell(t_0)/\partial t$ and, consequently, that $V_k(t) > V_\ell(t)$: all buyer types higher than t_0 will then strictly prefer to bid on seller *k*'s auction. Thus, it must be $t_1 = 1$ and $s_k^*(t) = 1$ for all $t \in (t_0, 1]$.

Part 2.

Assume that there exists some $t_1 > t_0 \geq 0$ such that $s_k^*(t) > 0$ for $t \leq t_0$ and $s_k^*(t) = 0$ for $t \in (t_0, t_1]$. According to Part 1 of Proposition 2, for $t \leq t_0$ it cannot be $s_k^*(t) = 1$ (otherwise, it would have to be $s_k^*(t) = 1$ for all $t > t_0$). Thus, for $t \leq t_0$ the equilibrium has to be mixed. At $t = t_0$ it must then be $V_k(t_0) = V_\ell(t_0)$ (equilibrium is mixed) and $\partial V_k(t_0)/\partial t \leq \partial V_\ell(t_0)/\partial t$ for at least one $\ell \neq k$, or equivalently:

$$r_k\big(1 - Q_k(1) + Q_k(t_0)\big)^{N-1} \leq r_\ell\big(1 - Q_\ell(1) + Q_\ell(t_0)\big)^{N-1}$$

The assumption $s_k^*(t) = 0$ for $t \in (t_0, t_1]$ implies $Q_\ell(t) > Q_\ell(t_0)$ and $Q_k(t) = Q_k(t_0)$ for all $t \in (t_0, t_1]$. But this, in turn, implies that $\partial V_\ell(t)/\partial t > \partial V_\ell(t_0)/\partial t \geq \partial V_k(t_0)/\partial t$ and, consequently, that $V_\ell(t) > V_k(t)$: no buyer type higher than t_0 will then prefer to bid on seller *k*'s auction. Thus, it must be $t_1 = 1$ and $s_k^*(t) = 0$ for all $t \in (t_0, 1]$.

Part 3.

Suppose that there exists some $t_0 \geq 0$ such that $s_k^*(t) > 0$ for $t \leq t_0$ and $s_k^*(t) = 0$ for $t > t_0$. It must then be $Q_k(t_0) = Q_k(1)$. Assume now that there exists some $\ell > k$, such that $s_\ell^*(t) > 0$ for $t > t_0$. It must be $Q_\ell(t_0) < Q_\ell(1)$. Furthermore, our convention for ordering sellers implies that $r_\ell \leq r_k$. At $t = t_0$ it must be $V_k(t_0) = V_\ell(t_0)$ (by Part 2, $s_\ell^*(t) > 0$ for $t > t_0$ implies $s_\ell^*(t) > 0$ for $t \leq t_0$) and $\partial V_k(t_0)/\partial t \leq \partial V_\ell(t_0)/\partial t$, or equivalently:

$$\begin{gathered} r_k\big(1 - Q_k(1) + Q_k(t_0)\big)^{N-1} \leq r_\ell\big(1 - Q_\ell(1) + Q_\ell(t_0)\big)^{N-1} \\ r_k\big(1 - Q_k(1) + Q_k(1)\big)^{N-1} \leq r_\ell\big(1 - Q_\ell(1) + Q_\ell(t_0)\big)^{N-1} \\ r_k < r_\ell \end{gathered}$$

The above contradicts the assumption $r_\ell \leq r_k$.

Part 4.

At the point $t_0 \geq 0$ where $s_k^*(t), s_\ell^*(t)$ switch from positive values to zero it must be $\partial V_k(t_0)/\partial t = \partial V_\ell(t_0)/\partial t$. In addition, it must be $Q_k(t_0) = Q_k(1)$, and $Q_\ell(t_0) = Q_\ell(1)$. Substitution into (4) immediately produces the result.

Proposition 3

We will show that a given set of reputations $r_1 \leq ... \leq r_M$ defines a unique $L \leq M$, associated zone delimiters t_z, $z = 0,...,L$, and selection probabilities $s_k(t)$ that satisfy the following incentive compatibility constraints:

- **IC1:** $V_k(t) = V_z(t)$ for all $k = 1,..,z$ and $t \leq t_{z-1}$
- **IC2:** $V_k(t) > V_z(t)$ for all $k = 1,..,z-1$ and $t > t_{z-1}$

Since $V_k(0) = 0$ for all *k*, at $t = 0$ (IC1) implies that $\partial V_k(0)/\partial t = \partial V_\ell(0)/\partial t$ for all $k = 1,...,L$. If $L < M$ then it must also be $\partial V_k(0)/\partial t < \partial V_L(0)/\partial t$ for $k = L+1,..,M$. From $\frac{\partial V_k(t)}{\partial t} = r_k\left(1 - Q_k(1) + Q_k(t)\right)^{N-1}$ (IC1) then implies:

$$r_k\left(1 - Q_k(1)\right)^{N-1} = r_L\left(1 - Q_L(1)\right)^{N-1} \text{forall} 1 \leq k \leq L \tag{23}$$

Equation (23) plus the one-bid constraint $\sum_{k=1}^{L} Q_k(1) = 1$ give:

$$Q_k(1) = 1 - (L-1)\frac{\sqrt[N-1]{\frac{1}{r_k}}}{\sum_{i=1}^{L}\sqrt[N-1]{\frac{1}{r_i}}} \tag{24}$$

Since, for $k = L+1,..,M$ it is $Q_k(t) = 0$ for all *t*, (IC2) is equivalent to:

$$r_k < r_L\left(1 - Q_L(1)\right)^{N-1} \text{forall} L + 1 \leq k \leq M$$

Substituting (24), the above expression becomes:

$$r_k < \left(\frac{L-1}{\sum_{i=1}^{L} \sqrt[N-1]{\frac{1}{r_i}}} \right)^{N-1} \text{forall} L+1 \leq k \leq M$$

Since $r_{L+1} \geq r_{L+2} \geq ... \geq r_M$ the above inequality can be rewritten as:

$$r_{k+1} < \left(\frac{k-1}{\sum_{i=1}^{k} \sqrt[N-1]{\frac{1}{r_i}}} \right)^{N-1} \text{forsome} k < M \tag{25}$$

The number of equilibrium type zones *L* is equal to the lowest *k* that satisfies (25). If no $L < M$ satisfies (25) then $L = M$ and all sellers will receive bids with positive probability from at least some buyers.

Constraint (IC1) plus the fact that $V_k(0) = 0$ for all *k* implies that $\partial V_k(t)/\partial t = \partial V_z(t)/\partial t$ and $\partial^2 V_k(t)/\partial t^2 = \partial^2 V_z(t)/\partial t^2$ for all $k = 1,..,z$ and $t < t_{z-1}$. From $\frac{\partial V_k(t)}{\partial t} = r_k \left(1 - Q_k(1) + Q_k(t)\right)^{N-1}$, $\frac{\partial^2 V_k(t)}{\partial t^2} = r_k (N-1) \left(1 - Q_k(1) + Q_k(t)\right)^{N-2} s_k(t) f(t)$ we obtain:

$$s_k(t) = \sqrt[N-1]{\frac{r_z}{r_k}} s_z(t) \text{forall} k = 1,..,z, t < t_{z-1}$$

Together with the one-bid condition $\sum_{k=1}^{z} s_k(t) = 1$ this implies that buyers whose types fall in zone *z* select seller k with constant probability:

$$s_{zk} = \frac{\sqrt[N-1]{\frac{1}{r_k}}}{\sum_{i=1}^{z} \sqrt[N-1]{\frac{1}{r_i}}} \tag{26}$$

We will now prove by induction that equality of derivatives $\partial V_k(t)/\partial t$ and the condition $Q_z(t_{z-1}) = Q_z(1)$ at type delimiters $t_1,..,t_{L-1}$ uniquely defines those delimiters as the solution of the equation:

$$F(t_z)=\left(\sqrt[N-1]{r_{z+1}}\sum_{i=1}^{z}\sqrt[N-1]{\frac{1}{r_i}}\right)-(z-1) \tag{27}$$

6. At $t=t_1$, the conditions $\partial V_1(t)/\partial t=\partial V_2(t)/\partial t$ and $Q_2(t_1)=Q_2(1)$ imply that:

$$r_1\left(1-(1-F(t_1))\right)^{N-1}=r_2\left(1\right)^{N-1}$$

$$F(t_1)=\sqrt[N-1]{\frac{r_2}{r_1}}=\left(\sqrt[N-1]{r_2}\sqrt[N-1]{\frac{1}{r_1}}\right)-(1-1)$$

7.

8. Assume now that $F(t_y)=\left(\sqrt[N-1]{r_{y+1}}\sum_{i=1}^{y}\sqrt[N-1]{\frac{1}{r_i}}\right)-(y-1)$ for $y<z$. At $t=t_z$, the conditions $\partial V_z(t)/\partial t=\partial V_{z+1}(t)/\partial t$ and $Q_{z+1}(t_z)=Q_{z+1}(1)$ imply that:

$$r_z\left(1-s_{zz}(F(t_{z-1})-F(t_z))\right)^{N-1}=r_{z+1}\left(1\right)^{N-1}$$

$$r_z\left(1-\frac{\sqrt[N-1]{\frac{1}{r_z}}}{\sum_{i=1}^{z}\sqrt[N-1]{\frac{1}{r_i}}}\left(\left(\sqrt[N-1]{r_z}\sum_{i=1}^{z-1}\sqrt[N-1]{\frac{1}{r_i}}\right)-(z-2)-F(t_z)\right)\right)^{N-1}=r_{z+1}\left(1\right)^{N-1}$$

$$F(t_z)=\left(\sqrt[N-1]{r_{z+1}}\sum_{i=1}^{z}\sqrt[N-1]{\frac{1}{r_i}}\right)-(z-1)$$

9.

The requirement $F(t_z)>0$ for $z=1,\ldots,L-1$ implies that equilibria of the above form exist if and only if $\left(\sqrt[N-1]{r_{z+1}}\sum_{i=1}^{z}\sqrt[N-1]{\frac{1}{r_i}}\right)-(z-1)$ >0 or, equivalently, if $r_{z+1}>\left(\frac{z-1}{\sum_{i=1}^{z}\sqrt[N-1]{\frac{1}{r_i}}}\right)^{N-1}$ for all $1<z\le L-1$. This is consistent with the definition of *L* as the lowest integer *k* for which $r_{k+1}<\left(\frac{k-1}{\sum_{i=1}^{k}\sqrt[N-1]{\frac{1}{r_i}}}\right)^{N-1}$.

It is easy to show that the above strategy profile also satisfies constraint (IC2). From

$$\frac{\partial V_z(t)}{\partial t}=r_z\left(1-Q_z(1)+Q_z(t)\right)^{N-1}$$

it follows that $\frac{\partial V_z(t)}{\partial t}$ increases with *t* for $t\le t_{z-1}$ and then remains constant and equal to r_z for $t>t_{z-1}$. Let us now consider some $k<z$. The assumption of a mixed equilibrium implies $\frac{\partial V_z(t)}{\partial t}=\frac{\partial V_k(t)}{\partial t}$ for all

$t \le t_{z-1}$. At $t = t_{z-1}$, $\frac{\partial V_z(t)}{\partial t}$ turns into a constant whereas $\frac{\partial V_k(t)}{\partial t}$ keeps growing. Therefore, $\frac{\partial V_z(t)}{\partial t} < \frac{\partial V_k(t)}{\partial t} \Rightarrow V_z(t) < V_k(t)$ for all $k < z$ and all $t > t_{z-1}$.

From Lemma 1, the expected number of bids in each seller's auction is equal to $NQ_k(1)$. Substitution of equation (24) immediately proves Part 5 of the Theorem.

Proposition 4

Let $R_k(t) = 1 - Q_k(1) + Q_k(t)$. It is $0 \le R_k(t) \le 1$ and $R_k(1) = 1$ for all *t*. Equation (10) can then be rewritten as:

$$H_{2,k}^{\{r_1,\ldots,r_M\}} = 1 + \int_0^1 \left[(N-1)\left(R_k(t)\right)^N - N\left(R_k(t)\right)^{N-1} \right] dt$$

Differentiating $(N-1)\left(f\right)^N - N\left(f\right)^{N-1}$ with respect to *f* we find that, for $0 \le f < 1$, the derivative is negative. This means that the integrand $(N-1)\left(R_k(t)\right)^N - N\left(R_k(t)\right)^{N-1}$ is a monotonically declining function of $R_k(t)$. But this also means that, given two functions $R_k(t)$, $R_\ell(t)$:

- **(R):** If $R_k(t) \le R_\ell(t)$ for all $0 \le t \le 1$ and $R_k(t) < R_\ell(t)$ for at least some *t* then $H_{2,k}^{\{r_1,\ldots,r_M\}} > H_{2,\ell}^{\{r_1,\ldots,r_M\}}$

An equivalent way of stating result (R) is that if $R_k \succ_{FOSD} R_\ell$ (where $\succ_{FOSD}$ denotes strict first-order stochastic dominance ordering) then $H_{2,k}^{\{r_1,\ldots,r_M\}} > H_{2,\ell}^{\{r_1,\ldots,r_M\}}$.

We will apply the above result to show that, given $M-1$ other sellers with fixed reputations $r_1 \ge r_2 \ge \ldots \ge r_{M-1}$, the expected value of seller *k*'s second highest bidder is an increasing function of his reputation *r*. Consider the function $R_k(t) = 1 - Q_k(1) + Q_k(t)$. From Proposition 3, fixing the reputations of all other sellers, $Q_k(1)$ is an increasing function of seller *k*'s reputation (so $1 - Q_k(1)$ is a decreasing function of seller *k*'s reputation). Furthermore, all auction selection probabilities s_{zk} are decreasing functions of seller *k*'s reputation. Therefore, for all *t* for which seller *k* is considered by buyers with positive probability, $Q_k(t)$, and thus $R_k(t)$, are decreasing functions of seller *k*'s reputation. When *t* surpasses the threshold buyer type t_k above which seller *k* is no longer considered, then $Q_k(t) = Q_k(1)$, and thus $R_k(t)$ attains its maximum value 1 and stays constant thereafter. From Proposition 3, threshold t_k is an increasing function of seller *k*'s reputation. Taken together, the previous properties imply that, as seller *k*'s reputation increases, function $R_k(t)$ decreases for small *t* and attains unity later. Thus, seller reputations generate a set of functions R_k that are monotonically ordered according to strict FOSD. According to result (R) this implies that the expected value of a seller's second highest bidder is a monotonically increasing function of that seller's reputation.

A similar approach can be used to prove Part 2 of this Proposition. The fundamental observation (again, from Proposition 3) is that if the reputation of seller *k* remains fixed but the reputation of some other seller increases then $Q_k(1)$ decreases and all auction selection probabilities s_{zk} increase.

Proposition 5

Consider a bid vector $\mathbf{b}(t)$ where $b_k(t)=0$. We will show that there exists an $\varepsilon>0$, such that bidder $t\in(0,1)$ can increase her expected surplus by setting $b_k(t)=\varepsilon$. From (12) the probability of obtaining the item from at least one seller if one bids in all auctions except k is:

$$1-\prod_{j\in\{1,\ldots,M\}-\{k\}}(1-r_jG_j(b_j(t)))$$

Therefore, the marginal probability of obtaining the item from at least one seller if one places a non-zero bid in action k is equal to:

$$\left[1-(1-r_kG_k(b_k(t)))\prod_{j\in\{1,\ldots,M\}-\{k\}}(1-r_jG_j(b_j(t)))\right]-\left[1-\prod_{j\in\{1,\ldots,M\}-\{k\}}(1-r_jG_j(b_j(t)))\right]$$
$$=r_kG_k(b_k(t))\prod_{j\in\{1,\ldots,M\}-\{k\}}(1-r_jG_j(b_j(t)))$$

The incremental cost of participating on auction k is $\int_0^{b_k(t)}xg_k(x)dx$. The marginal expected surplus from participating in auction k is, thus, equal to:

$$\Delta V=tr_kG_k(b_k(t))\prod_{j\in\{1,\ldots,M\}-\{k\}}(1-r_jG_j(b_j(t)))-\int_0^{b_k(t)}xg_k(x)dx \tag{28}$$

Integration by parts gives $\int_0^{b_k(t)} xg_k(x)dx = b_k(t)G_k(b_k(t)) - \int_0^{b_k(t)} G_k(x)dx$. Substituting into (28) we obtain:

$$\Delta V = G_k(b_k(t))\left[tr_k \prod_{j\in\{1,\ldots,M\}-\{k\}} (1 - r_j G_j(b_j(t))) - b_k(t)\right] + \int_0^{b_k(t)} G_k(x)dx \tag{29}$$

If $r_j < 1$ it is $1 - r_j G_j(b_j(t)) > 0$. In the special case $r_j = 1$ the one-bid equilibrium (Proposition 3) shows that the top buyer type ($t = 1$) bids her expected valuation $b_j(1) = 1$ on auction j with probability 1 and also wins the auction with probability 1. This implies that $G_j(b_j(1)) = 1$. When $r_j = 1$ and $t = 1$ this behavior also holds in settings where the number of bids is unrestricted: since $1 - r_j G_j(b_j(1)) = 0$, maximization of equation (29) implies that the top type bids zero on all auctions $k \neq j$. But if type $t = 1$ bids $b_j(1) = 1$ on auction j, for any $t < 1$ it is $b_j(t) \leq t < 1$ and thus $G_j(b_j(t)) < G_j(1) = 1$. Hence, $1 - r_j G_j(b_j(t)) > 0$ even when $r_j = 1$.

If $1 - r_j G_j(b_j(t)) > 0$ for all j, for $t \in (0,1)$ it is also $tr_k \prod_{j\in\{1,\ldots,M\}-\{k\}} (1 - r_j G_j(b_j(t))) > 0$. Then, for any $0 < \varepsilon < tr_k \prod_{j\in\{1,\ldots,M\}-\{k\}} (1 - r_j G_j(b_j(t)))$ setting $b_k(t) = \varepsilon$ makes both terms of (29) strictly positive and thus strictly increases the bidder's expected surplus.