Capitalizing Implementation Cycles *

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(Work in progress)

June 19, 2009

Abstract

Building on Shleifer (1986), I introduce capital to a world with no uncertainty, where agents share the same expectations about future and have perfect foresight. I subsequently show that, although firms acquire patents at an exogenous, perfectly smooth rate, they may coordinate and implement these simultaneously if their expectations are accordingly formed, due to the presence of strategic complementarities. Then the economy can still grow in cycles, although investment is fully reversible and a storable commodity present, and, despite agents being able to perfectly borrow against their future earnings. This permits the study of the evolution of aggregate economic magnitudes along the balanced growth path. It is shown that consumption and investment fluctuate relative to trend, and may fluctuate even in absolute terms, despite the economy being subject to 'good' only, idiosyncratic productivity shocks.

JEL Classification: D42, D51, E21, E22, E32.
Keywords: Business cycle, aggregate fluctuations, capital, innovation, monopoly, multiple equilibria.

*I feel grateful to Herakles Polemarchakis for his excellent guidance and his invaluable suggestions and support, and to Paulo Santos Monteiro for the many fruitful discussions we have had. I would also like to thank Atsuyoshi Morozumi for providing me with useful comments at various stages of this work, as well as seminar participants at the Theory Workshop at the University of Warwick. Financial support from the ESRC and the George and Victoria Karelias Foundation is gratefully acknowledged. Correspondence: Michail Rousakis, Department of Economics, *Room S.113b*, University of Warwick, Coventry, CV4 7AL. E-mail address: M.Rousakis@warwick.ac.uk

1 Introduction

It has been argued [1] that in a world free from uncertainty, where agents are endowed with perfect-foresight and share the same expectations about future, the economy cannot cycle in the presence of capital, or any storable commodity, unless borrowing constraints or investment irreversibility are present; consumption smoothing would render cycles impossible. Indeed, I show here that cycles can well take place in the presence of capital, without imposing any of these constraints.

The baseline model in such an attempt is Shleifer (1986) which introduced implementation cycles in an economy with only labor present and any storable commodity absent. This paper builds on it. The central difference between this framework and the current one, is that here I allow for capital and savings, and, contrary to Shleifer's conjecture, implementation cycles can be still generated.

Model. The economy consists of a final-good representative firm, two sectors with each comprising a high enough number of capital (intermediate) -good firms and a representative agent owning all firms in the economy. In contrast with Shleifer (1986), the innovation takes place in the capital-good sectors, rather than the final-good sector.[2] The capital-good firms use the unconsumed final-good as input today to produce a capital-good tomorrow. Capital-good firms are not price-takers, and can profit by implementing a patent (*idea*) which offers them the advantage over the other firms in their sector, to produce the capital-good at a lower cost; this inherent non-price-taking behavior implies the economy can reach at most a second best.

Like in Shleifer (1986), patents arrive like manna from heaven at a constant, perfectly smooth rate, and they are idiosyncratic to a firm in each sector. Unlike in Romer (1990), firms can profit out of these only once, as in the period following implementation imitators will copy the patents and profits will be eliminated. Therefore, firms may find it optimal not to immediately implement their patents, but instead to 'save' these for the future and implement these simultaneously.

Strategic complementarities and common expectations are key to such an outcome. The backbone of the model, in line with Shleifer (1986), consists of these, along with the previously described one-off profits. The former implies that higher profits in a capital-sector are translated into higher demand for the product of the other ones, rendering thus delay in implementation potentially profitable. The latter is the often articulated concept of 'animal spirits', following from Keynes (1936). Agents form expectations arbitrarily of the timing of the patents' implementations.

These two in tandem imply that our economy admits multiple equilibria.[3] Throughout the focus will be on perfect-foresight, periodic and stationary equilibria. Any strategic interaction

[1] See for example Shleifer (1986) and Francois and Lloyd-Ellis (2008).

[2] The structure of our economy, save the research sector, resembles the one in Romer (1990).

[3] Related papers exhibiting these features, but in an overlapping-generations economy, are Azariadis (1981) and Grandmont (1985).

among firms will be assumed away.

Preview of results. The central thesis of the paper is that implementation cycles can be generated in the presence of capital, which is storable. This happens although investment is fully reversible and agents face no borrowing constraints. To the best of my knowledge this result is new. I show that this can happen for sufficiently patient agents, great enough innovations and low enough curvature of the utility function. These comparative statics are similar to the ones in Shleifer (1986). Note though, that all effects must be mild enough, for otherwise the transversality condition will be violated. Crucial to the development of the results is that the representative agent be a price-taker. Were things viewed from a social planner's point of view, the economy might not grow in cycles. Indeed, I show that for logarithmic preferences cycles are Pareto inferior compared to the steady-growth equilibrium. The latter is shown to be always attainable.

Allowing for capital and savings enables the analysis to shed light on the evolution of output, consumption and investment in such a framework. It is shown below that, controlling for departures from trend, consumption is procyclical, while investment is countercyclical. Investment is more volatile than consumption and output, whereas for sufficiently high concavity of agent's preferences, or which is the same here, for sufficiently low intertemporal elasticity of substitution, and sufficiently low innovation rate, consumption will be more volatile than output, and, may even overshoot. Higher concavity in preferences is translated into, in general, lower volatility in the aggregate variables.

Other Related Literature. By permitting the study of the evolution of aggregate magnitudes the cycle, this paper brings closer Shleifer (1986) with the Real Business Cycle theory, introduced in Kydland and Prescott (1982). However, there are important remaining differences between them. First, in the RBC models cycles are Pareto optimal, whereas here, due to the temporary monopoly power in the capital-goods sector, they can at most be a second-best outcome. Indeed, I show that for logarithmic preferences, they are Pareto inferior (third-best) to the acyclical equilibrium. Second, I abstract from employment considerations and a time-to-build technology; indeed here capital takes only one period to be built. Models share that productivity (real) shocks hit the economy. In the RBC framework it is the way exogenous shocks hit the economy that generates the cycles. On the contrary, here shocks are fully anticipated and agents are free to borrow against their future profits. Cycles are sparked by expectations, reminding us of what Keynes (1936) labeled as 'animal spirits', which go hand-in-hand with the profits that capital-good firms enjoy due to these, temporarily idiosyncratic, but fully-anticipated shocks. A strength of this paper and Shleifer (1986) is that in spite of patents reaching the economy at an exogenously determined, perfectly smooth rate, yet they may be implemented in a cyclical fashion; put it otherwise, although from an economy point of view there is a constant flaw of patents hitting the capital-good sectors, yet it may be in their interest not to implement these outright, triggering thereby implementation cycles. Were shocks unanticipated or/and uninsurable, the described mechanism would amplify the above effects. Last, this paper differentiates itself from Kydland and Prescott (1982) by employing a time-separable utility function and requiring more that one types of capital to be used in the production of the final-good.

Francois and Lloyd-Ellis (2008) introduce capital to a framework based on Shleifer (1986), but they endogenize the way patents arrive, which drives their result: patents arriving in cycles are more easily implemented in cycles too. A good feature of their environment, though, is that it admits a unique equilibrium.

Last, this work shares with Shleifer (1986) the fact that it casts doubt on the perception that idiosyncratic to the firm level shocks cannot necessarily cause aggregate fluctuations, a view appearing in Lucas (1987).

Structure. The rest of the paper is organized as follows. In Section 2 the model is described. Section 3 presents the equilibria of an economy involving 2 capital-good sectors, and analyzes the comparative statics, as well as the evolution of the aggregate variables along the balanced growth path. Section 4 pursues a welfare comparison between the equilibria. Section 5 extends the analysis to an N capital-good sector economy. Section 6 suggests how this work will be continued.

2 Model

Environment. The decentralized economy is populated by an infinitely-lived representative agent. The representative agent consumes a consumption (final) good, produced by a final-good firm, against which he supplies inelastically his time endowment. The final-good firm also rents and uses in an additively-separable way two distinct types of capital. This is extended to N types in section 5. For each type of capital there is a respective sector comprising at least two firms which Bertrand-compete for the production of the capital-good which takes one period to be built. Foregone consumption (investment) is the input used in both capital-good sectors. All firms in the economy are owned by the representative agent.

There is no uncertainty, agents and firms have perfect foresight and common expectations. Capital markets are perfect and capital depreciates fully within a period. Time is discrete.

2.1 Representative Agent

The representative agent has lifetime utility:

$$\sum_{t=1}^{\infty} \rho^{t-1} \frac{{x_t}^{1-\gamma}}{1-\gamma}, \tag{1}$$

where $x_t \geq 0$ is the amount of final-good consumed, ρ is the discount factor, with $0 < \rho < 1$ and $\gamma > 0$ is the curvature of the utility function.

The representative agent is endowed with L units of time, which he supplies inelastically to the final-good firm. He is the unique shareholder of all firms in the economy. His lifetime budget constraint is then:

$$\sum_{t=1}^{\infty} D_t x_t \leq \sum_{t=1}^{\infty} D_t (w_t L + \Pi_{F,t} + \Pi_{C,t}), \tag{2}$$

where $D_t = \frac{1}{R_1 R_2 \ldots R_{t-1}}$ and $D_1 = 1$. R_t is the real interest rate paid in $t+1$, expressed in terms of the final-good ($R_{t-1} \equiv 1 + r_{t-1}$), and w_t is the real wage paid by the final-good firm; $\Pi_{F,t}$ and $\Pi C, t$ are the profits that accrue to the agent by the final-good and the capital-good firms respectively.

His decision problem is to choose $\{L_t^s, x_t\}_{t=1}^{\infty}$ to maximize his lifetime utility (1) subject to his lifetime budget constraint (2). The First Order Conditions with respect to x_{t-1}, x_t, x_{t+1} imply

$$\rho^{-2} \left(\frac{x_{t+1}}{x_{t-1}} \right)^{\gamma} = R_t R_{t-1}, \tag{3}$$

$$\rho^{-1} \left(\frac{x_{t+1}}{x_t} \right)^{\gamma} = R_t. \tag{4}$$

2.2 Final-Good Firm

The final-good firm is a price-taker in both the consumption-good and the factor-good markets. Its neoclassical technology is given by

$$F_t(L, k_1, k_2) = L_t^{\alpha} (k_{t,1}^{1-\alpha} + k_{t,2}^{1-\alpha}),^{4} \tag{5}$$

where L_t is the amount of labor employed, $k_{t,1}$ and $k_{t,2}$ are the amounts of capital types 1 and 2, respectively, rented [5] in period t, and $\alpha \in (\frac{1}{2}, 1)$ is the labor share parameter.

[4]The same production funcion can be found in Romer (1990).

[5]Final-good firms are assumed to rent rather than buy capital goods, and thus I circumvent a possible durable-goods monopoly problem.

Note that the marginal products of capital are independent, whereas the intratemporal elasticity of substitution between the two types of capital is $1/\alpha$. Full capital depreciation is assumed with no loss of generality.

After normalizing with respect to the consumption good, the firm chooses $\{L_t^d, k_{t,1}^d, k_{t,2}^d, y_t^s, i_t^s\}_{t=1}^{\infty}$ to maximize its period t (real) profits given by

$$\Pi_{F,t} = L_t^{\alpha}(k_{t,1}^{1-\alpha} + k_{t,2}^{1-\alpha}) - w_t L_t - q_{t,1}k_{t,1} - q_{t,2}k_{t,2},$$

where w_t is the wage paid to labor denominated in units of the consumption good, and $q_{t,i}$ is the (real) rental price of capital type i, where $i = 1, 2$, and $y_t = F_t(L, k_1, k_2)$.

The firm's maximization problem yields

$$w_t = \alpha L_t^{\alpha-1}(k_{t,1}^{1-\alpha} + k_{t,2}^{1-\alpha}), \tag{6}$$

and

$$q_{t,i} = (1-\alpha)L_t^{\alpha}k_{t,i}^{-\alpha}. \tag{7}$$

Constant returns to scale imply $\Pi_{F,t} = 0$ in every t. Check also that $\epsilon_{q,k_i} = \frac{1}{\alpha}$ and $\epsilon_{w,L} = \frac{1}{1-\alpha}$, and as $a \in (0, 1)$ both demands are elastic.

2.3 Capital-Good Firms

For each capital-good there is a class of a high enough number of capital makers who Bertrand-compete for its production. In case of a tie, a firm is randomly selected. The technology of capital maker j, where $j = 1, 2, \ldots J$ and J sufficiently high (see also fn. 6) is

$$k_{t,i} = n_{t-1,i}i_{t-1,i}, \tag{8}$$

where $k_{t,i}$ is capital type i produced by a capital maker in sector i, where $i = 1, 2$, in period t, $i_{t-1,i}$ is demand for investment, defined as the foregone consumption in $t-1$, which is input in the production of capital.

Investment is fully reversible, or as in Romer (1990), capital-goods are putty-putty. This is innocuous, however, as in the stationary equilibrium demand for capital-goods is positive and no disinvestment happens.

It follows that demand for investment in t is

$$i_t = i_{t,1} + i_{t,2}. \tag{9}$$

The economy starts with an initial level of investment depending on the capital needs of period 1. That is,

$$i_0 = \frac{k_{1,1}}{n_{1,1}} + \frac{k_{1,2}}{n_{1,2}}. \tag{10}$$

Capital-good firms choose $\{k^s_{t,1}, k^s_{t,2}, i^d_t\}_{t=1}^{\infty}$ to maximize their profits provided below. Investment is fully reversible.

Pattern of Patents. Like in Shleifer (1986) patents arrive in an exogenously determined order, like manna from heaven. Conventionally, a patent randomly reaches a firm in capital sector 1 in odd periods, and a firm in capital sector 2 in the even ones. [6] They affect capital-good i's technology in the following way:

$$\frac{n_{t+1,i}}{n_{t-1,i}} = \mu,$$

where $\mu > 1$. Suppose also that $n_{0,1} = 1$ and $n_{0,2} = \mu^{\frac{1}{2}}$, which implies that the capital sectors are symmetric. Note that there is no first-mover advantage, as the sectors alternate in leading state-of-the-art technology. Effectively, it is as if time were starting in $-\infty$.

A firm can profit only once [7] for each patent it acquires. In the period following the implementation of the patent, imitators enter the capital market and drive prices down to the marginal cost and profits to zero. As capital takes *one* period to build, implementing a patent in $t-1$ results in profit-making in t. Profits of capital-maker j are given by

$$\Pi_{t,i} = q_{t,i}k_{t,i} - R_{t-1}i_{t-1,i}.$$

As i_{t-1} is given by (9), profits boil down to

$$\Pi_{t,i} = q_{t,i}k_{t,i} - R_{t-1}\frac{k_{t,i}}{n_{t-1,i,j}},$$

where $\Pi_{t,i}$ denotes profits in sector i, $q_{t,i}$ is the rental price (inverse demand) of capital-good i expressed in terms of the consumption good given by (7), and $n_{t-1,i,j}$ is the inverse marginal cost of capital-maker j producing durable i. Monopoly profits will be an inherent source of inefficiency throughout the analysis.

[6] The number of firms, J, in a sector must be high enough that the probability of an idea reaching a firm is infinitesimally small.

[7] The assumption is based on Shleifer (1986) and it comes contrary to Romer (1990) where firms can profit forever.

Perfect Competition. Perfect competition in t implies $n_{t,i,j} = n_{t,i,-j}$, where $n_{t,i,-j}$ refers to the inverse marginal cost of the competitors' of agent j. Then $\Pi_{t,i} = 0$ and a randomly selected, with no loss of generality, firm in sector i sells the capital good i at a price

$$q_{t,i}^{*} = \frac{R_{t-1}}{n_{t-1,i,-j}}. \tag{11}$$

Inverse demand from the final-good firm (7) is taken as given and pins down the competitive quantity,

$$k_{t,i}^{*} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L_t n_{t-1,i,-j}^{\frac{1}{\alpha}}}{R_{t-1}^{\frac{1}{\alpha}}}. \tag{12}$$

Monopoly. A firm j in sector i implementing a patent in $t-1$ can enjoy monopoly profits periods for one period. In that case the capital maker j, who takes the inverse demand for capital (7) as given, chooses $k_{t,i}$ to solve:

$$\Pi_{t,i} = q_{t,i} k_{t,i} - R_{t-1} \frac{k_{t,i}}{\mu n_{t-1,i,-j}}.$$

This results in

$$q_{t,i}^{m} = \frac{R_{t-1}}{(1-\alpha)\mu n_{t-1,i,-j}},$$

and

$$k_{t,i}^{m} = \frac{(1-\alpha)^{\frac{2}{\alpha}} L_t (\mu n_{t-1,i,-j})^{\frac{1}{\alpha}}}{R_{t-1}^{\frac{1}{\alpha}}}.$$

But a monopolist cannot charge more than $q_{t,i}^{*}$, else his competitors will capture the market. In case of a tie, he can always set a price lower by ϵ than his competitors' and satisfy the whole demand. Recall also that the profit function is concave in k and q is decreasing in k and that demand for capital is elastic. Then, $q_{t,i}^{m} \geq q_{t,i}^{*}$ implies

$$\mu(1-\alpha) \leq 1. \tag{13}$$

The lower the innovation rate, μ, is, and the less elastic the demand for capital is (a greater α), the more easily this condition is satisfied. Hereafter, I will restrict attention to the case where (13) holds. Then the monopolist sets the competitive price (11) and produces the competitive quantity (12), enjoying profits due to his lower by μ cost of producing a unit of capital. These are given by

$$\Pi_{t,i} = q_{t,i}^{*} k_{t,i}^{*} - R_{t-1} \frac{k_{t,i}^{*}}{\mu n_{t-1,i,-j}},$$

which combined with (11) and (12) become

$$\Pi_{t,i} = (1-\alpha)^{\frac{1}{\alpha}} L_t n_{t-1,i,-j}^{\frac{1}{\alpha}-1} \frac{1}{R_{t-1}^{\frac{1}{\alpha}-1}} \left(\frac{\mu-1}{\mu}\right). \tag{14}$$

It becomes evident from the above that profits depend inversely on the interest rate of the period preceding the one profits are made; a higher interest yesterday results in less capital produced today, and, consequently, in lower profits for the monopolist.

2.3.1 Deciding to Implement

As it takes one period for capital to build, implementing an idea today results in profiting tomorrow. A firm in sector i receiving a patent in $t-1$ will implement immediately if and only if (superscripts denote the date the patent arrives, subscripts the date profits are made, which as claimed is one period after implementation):

$$\frac{\Pi_{t,i}^{t-1}}{R_{t-1}} \geq \frac{\Pi_{t+1,i}^{t-1}}{R_{t-1}R_t}.$$

By (14), this boils down to

$$\left(\frac{R_t}{R_{t-1}}\right)^{\frac{1}{\alpha}-1} \geq \frac{1}{R_t}. \tag{15}$$

Needless to say, a non-satisfied (15) is a necessary condition for a synchronized (*cyclical*) equilibrium to be sustained. Notice that, although profits depend on the competitors' technology level at the period of the implementation, the decision to implement is independent of it.

3 Balanced Growth Path Equilibrium

Below I shed light on the equilibria of our economy, depending on agents' expectations. It is crucial for our results that firms share the same expectations and are endowed with perfect foresight, with these being common knowledge among them. First, it is shown that, should immediate implementation be anticipated, the *acyclical* equilibrium is always sustained, and, next, for which values of the parameters $\{\alpha, \gamma, \mu, \rho\}$ a synchronized (*cyclical*) equilibrium is realized, when firms' expectations are such.

Stationarity. My focus is on stationary equilibria, that is equilibria whose pattern is independent of time; the analysis will be restricted to the balanced growth path. I impose the following restrictions:

$$\frac{x_{t+1}}{x_{t-1}} = \lambda\,, \tag{16}$$

and

$$\frac{x_{t+1}}{x_t} = v > 0^8. \tag{17}$$

Combined with (3) and (4) these imply

$$R_t = \begin{cases} \frac{v^\gamma}{\rho}, & \text{if } t \text{ even} \\ \left(\frac{\lambda}{v}\right)^\gamma \frac{1}{\rho}, & \text{if } t \text{ odd} \end{cases}. \tag{18}$$

That interest rates are repeated every two periods is key to our results.

Remark 1. *Check that* $R_t R_{t+1} = \frac{\lambda^\gamma}{\rho^2}$.

Time and Symmetry. Patents in sector 1 arrive in odd periods, while in sector 2 in the even ones. Henceforth, period τ is an even period. As the economy was shown to be stationary, attention will be restricted to just period τ and the periods before and after it. As I want sectors to be symmetric I further suppose that when in $\tau - 1$ an innovation arrives to a firm in sector 1, the then up-to-date technology, irrespectively of whether it is implemented or not, becomes μn, whereas in sector 2 it remains $\mu^{1/2} n$; in τ an innovation reaches a firm in sector 2 where the up-to-date technology then becomes $\mu^{3/2} n$, whereas in sector 1 it remains μn; thereby the ratio of leading technologies in the two sectors equals $\mu^{1/2}$ in $\tau - 1$ and $\frac{1}{\mu^{1/2}}$ in τ and this pattern continues in the following periods with the sectors alternating in leading the patent race and no firm having a first-mover advantage. This further renders the analysis independent of when our world starts; effectively, it as if time were starting in $-\infty$.

Market-Clearing. In equilibrium markets must clear which implies $i_t^d = i_t^s \equiv i_t$, $k_{t,i}^d = k_{t,i}^s \equiv k_{t,i}$, $L_t^d = L_t^s = L$, and $y_t = x_t + i_t$, for each t.

It will be the standard practice hereafter, to customize the analysis to the two different equilibria which can prevail in our economy: the immediate implementation (*acyclical*) equilibrium and the synchronized implementation (*cyclical*) one.

3.1 Immediate Implementation

3.1.1 Demands and Supplies

Period τ-1. An idea arrives to sector 1 and is immediately implemented by its recipient. As (13) is assumed to hold throughout, the competitive price (11) is set and the competitive quantity (12)

[8] A negative v, combined with the fact that $\lambda > 0$ as will be shown below, would imply negative gross interest rates, and, consequently negative invesment, consumption and production. Therefore, I rule it out.

is produced. In line with the above, technology levels in the two sectors are $n_{\tau-1,1,-j} = n_{\tau-2,1} = n < n_{\tau-1,1,j} = \mu n$, as $\mu > 1$, and $n_{\tau-1,2,-j} = n_{\tau-1,2,j} = n_{\tau-2,2} = \mu^{\frac{1}{2}} n$. Then,

$$k_{\tau,1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}}}{R_{\tau-1}^{\frac{1}{\alpha}}}, \tag{19}$$

and

$$k_{\tau,2} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L \left(n\mu^{\frac{1}{2}}\right)^{\frac{1}{\alpha}}}{R_{\tau-1}^{\frac{1}{\alpha}}}. \tag{20}$$

Then as by (9) $i_{\tau-1} = i_{\tau-1,1} + i_{\tau-1,2}$, we get that $i_{\tau-1} = \frac{k_{\tau,1}}{\mu n} + \frac{k_{\tau,2}}{\mu^{\frac{1}{2}} n}$. Combined with (19) and (20),

$$i_{\tau-1} = \left(\frac{1}{\mu} + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}\right) \frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1}}{R_{\tau-1}^{\frac{1}{\alpha}}}. \tag{21}$$

Check that investment decreases in the interest rate.

Similarly as $n_{t-2,1,-j} = n$ and $n_{t-2,2,-j} = \frac{n}{\mu^{\frac{1}{2}}}$,

$$k_{\tau-1,1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}}}{R_{\tau-2}^{\frac{1}{\alpha}}}. \tag{22}$$

and

$$k_{\tau-1,2} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L \left(\frac{n}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}}}{(R_{\tau-2})^{\frac{1}{\alpha}}}. \tag{23}$$

Combining (5), with (22) and (23),

$$y_{\tau-1} = \left(1 + \left(\frac{1}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}-1}\right) (1-\alpha)^{\frac{1}{\alpha}-1} L n^{\frac{1}{\alpha}-1} \frac{1}{R_{\tau-2}^{\frac{1}{\alpha}-1}}.$$

Note that output decreases in last period's interest rate; a higher interest rate today results in less investment today, and, thus, in lower production tomorrow.

From the market clearing condition for the consumption good

$$y_{\tau-1} = x_{\tau-1} + i_{\tau-1},$$

$x_{\tau-1}$ can be pinned down. It follows then that

$$x_{\tau-1} = (1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1} \left(\frac{1 + \left(\frac{1}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}-1}}{(1-\alpha) R_{\tau-2}^{\frac{1}{\alpha}-1}} - \frac{\frac{1}{\mu} + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}}{R_{\tau-1}^{\frac{1}{\alpha}}} \right). \tag{24}$$

The lower yesterday's interest rate, the higher today's production, and the higher today's interest rate, the lower today's investment, which result in higher consumption today.

Period τ. An idea arrives to sector 2 and is immediately implemented by its recipient. In τ technology in the two sectors is $n_{\tau,1,-j} = n_{\tau,1,j} = \mu n$ and $n_{\tau,2,-j} = \mu^{\frac{1}{2}} n < n_{\tau,2,j} = \mu^{\frac{3}{2}} n$. The thought process is like before. Then,

$$k_{\tau+1,1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu n)^{\frac{1}{\alpha}}}{R_{\tau}^{\frac{1}{\alpha}}}, \tag{25}$$

and

$$k_{\tau+1,2} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L\left(\mu^{\frac{1}{2}} n\right)^{\frac{1}{\alpha}}}{R_{\tau}^{\frac{1}{\alpha}}}. \tag{26}$$

Then $i_\tau = \frac{k_{\tau+1,1}}{\mu n} + \frac{k_{\tau+1,2}}{\mu^{\frac{3}{2}} n}$, which implies

$$i_\tau = \left(\mu^{\frac{1}{\alpha}-1} + \frac{\mu^{\frac{1}{2\alpha}}}{\mu^{\frac{3}{2}}}\right) \frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1}}{R_{\tau}^{\frac{1}{\alpha}}}.$$

To find output, we can plug (19) and (20) into (5) to get

$$y_\tau = (1 + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)})(1-\alpha)^{\frac{1}{\alpha}-1} L n^{\frac{1}{\alpha}-1} \frac{1}{R_{\tau}^{\frac{1}{\alpha}-1}}.$$

Market-clearing in the consumption-good market implies

$$x_\tau = (1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1} \left(\frac{1 + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}}{(1-\alpha) R_{\tau}^{\frac{1}{\alpha}-1}} - \frac{\mu^{\frac{1}{\alpha}-1} + \frac{\mu^{\frac{1}{2\alpha}}}{\mu^{\frac{3}{2}}}}{R_{\tau}^{\frac{1}{\alpha}}} \right). \tag{27}$$

Period τ+1. An idea arrives to sector 1 and is immediately implemented by its recipient. Technology in the two sectors is $n_{\tau+1,1,-j} = \mu n < n_{\tau+1,1,j} = \mu^2 n$ and $n_{\tau+1,2,-j} = n_{\tau+1,2,j} = \mu^{\frac{3}{2}} n$. Then,

$$k_{\tau+2,1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L \mu^{\frac{1}{\alpha}} n^{\frac{1}{\alpha}}}{R_{\tau+1}^{\frac{1}{\alpha}}}, \tag{28}$$

and

$$k_{\tau+2,2} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L \mu^{\frac{3}{2}(\frac{1}{\alpha})} n^{\frac{1}{\alpha}}}{R_{\tau+1}^{\frac{1}{\alpha}}}. \tag{29}$$

As $i_{\tau+1} = \frac{k_{\tau+2,1}}{\mu^2 n} + \frac{k_{\tau+2,2}}{\mu^{\frac{3}{2}} n}$, (28) and (29) imply

$$i_{\tau+1} = \left(\mu^{\frac{1}{\alpha}-2} + \mu^{\frac{3}{2}(\frac{1}{\alpha}-1)}\right) \frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1}}{R_{\tau+1}^{\frac{1}{\alpha}}}.$$

From (25), (26) and (5)

$$y_{\tau+1} = \left(\mu^{\frac{1}{\alpha}-1} + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}\right)(1-\alpha)^{\frac{1}{\alpha}-1} L n^{\frac{1}{\alpha}-1} \frac{1}{R_{\tau}^{\frac{1}{\alpha}-1}}.$$

Market-clearing in the consumption-good market implies

$$x_{\tau+1} = (1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1} \left(\frac{\mu^{\frac{1}{\alpha}-1} + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}}{(1-\alpha) R_{\tau}^{\frac{1}{\alpha}-1}} - \frac{\mu^{\frac{1}{\alpha}-2} + \mu^{\frac{3}{2}(\frac{1}{\alpha}-1)}}{R_{\tau+1}^{\frac{1}{\alpha}}} \right). \tag{30}$$

As $\alpha \in (\frac{1}{2}, 1)$, we can check that $x_t > 0$ for each t.

3.1.2 Equilibrium.

For an immediate implementation (*acyclical*) equilibrium to be sustained, no firm must find it optimal to postpone implementation to some period after the cycle. This implies (15) must hold for $t = \tau - 1, \tau$:

$$\left(\frac{R_{\tau}}{R_{\tau-1}} \right)^{\frac{1}{\alpha}-1} \geq \frac{1}{R_{\tau}},$$

and

$$\left(\frac{R_{\tau+1}}{R_{\tau}} \right)^{\frac{1}{\alpha}-1} \geq \frac{1}{R_{\tau+1}}.$$

The former refers to the firm in sector 1 receiving a patent in $\tau - 1$ and the latter to the firm in sector 2 receiving a patent in τ.

Furthermore, no firm must have an incentive to wait past the cycle, irrespectively of the fact that a new patent will arrive in the sector rendering the one in question obsolete. This argument, which will be extensively analyzed below, is also discussed in Shleifer (1986). That is (superscripts denote the date the patent arrives, subscripts the date profits are made which as claimed is one period after implementation),

$$\frac{\Pi_{\tau,i}^{\tau-1}}{R_{\tau-1}} \geq \frac{\Pi_{\tau+2,i}^{\tau-1}}{R_{\tau-1} R_{\tau} R_{\tau+1}},$$

and as by (18) $R_{\tau-1} = R_{\tau+1}$,

$$1 \geq \frac{1}{R_{\tau} R_{\tau+1}}. \tag{31}$$

Conditions (15) and (31) are necessary and sufficient for an immediate implementation equilibrium to be sustained, as long as they are true for each sector i in each period $\tau \in t$.

For $R_{\tau-1} = R_{\tau} \equiv R$, from (16), (24) and (30) we get that $\lambda = \mu^{(\frac{1}{\alpha}-1)}$, which combined with Remark 1 implies that (31) holds; given this, from (17) we find that $v = \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}$, which can be also confirmed from (27) and (30). Interestingly, both imply $\frac{x_{\tau+1}}{x_{\tau}} = \frac{x_{\tau}}{x_{\tau-1}} = \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}$, and by (18),

$$R = \frac{\mu^{\frac{\gamma}{2}(\frac{1}{\alpha}-1)}}{\rho}. \tag{32}$$

Notice that $R > 1$, as $\mu > 1$, $\gamma > 0$, $\alpha \in (\frac{1}{2}, 1)$ and $\rho \in (0, 1)$. Then, it is easy to check that (15) is always met.

Corollary 1. *An acyclical (steady-growth) equilibrium is always attainable.*

This presumes that the transversality condition (58) analyzed below holds.

Balanced Growth Path. As $\lambda = \mu^{\frac{1}{\alpha}-1}$, and given (32), we can see from the above that

$$\frac{y_{\tau+1}}{y_\tau} = \frac{y_\tau}{y_{\tau-1}} = \frac{i_{\tau+1}}{i_\tau} = \frac{i_\tau}{i_{\tau-1}} = \frac{x_{\tau+1}}{x_\tau} = \frac{x_\tau}{x_{\tau-1}} = \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}. \tag{33}$$

Remark 2. *For symmetric firms, in an acyclical equilibrium, output, consumption and investment grow at the same perfectly smooth rate.*

3.2 Synchronized Implementation

3.2.1 Demands and Supplies

The time pattern of the arrival of patents is as described above. In a synchronized (*cyclical*) implementation equilibrium a firm in sector 1 receiving a patent in an odd period must fund it optimal to implement in the following even period. This happens with no loss of generality, since as shown above, sectors are symmetric.

Period τ-1. A patent arrives to sector 1 and is not implemented by its recipient, but kept (*saved*) and implemented in τ. Effectively, from a $\tau - 1$ viewpoint, a not implemented patent is as if it had never arrived. As (13) holds, the competitive price (11) is set and the competitive quantity (12) is produced. The technology with which next period's capital will be produced is $n_{\tau-1,1,-j} = n_{\tau-1,1,j} = n_{\tau-2,1} = n$ and $n_{\tau-1,2,-j} = n_{\tau-1,2,j} = n_{\tau-2,2} = \mu^{\frac{1}{2}} n$. Then

$$k_{\tau,1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}}}{R_{\tau-1}^{\frac{1}{\alpha}}}, \tag{34}$$

and

$$k_{\tau,2} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L (\mu^{\frac{1}{2}} n)^{\frac{1}{\alpha}}}{R_{\tau-1}^{\frac{1}{\alpha}}}. \tag{35}$$

Then since $i_{\tau-1} = i_{\tau-1,1} + i_{\tau-1,2}$, we get that $i_{\tau-1} = \frac{k_{\tau,1}}{n} + \frac{k_{\tau,2}}{\mu^{\frac{1}{2}} n}$. Combined with (34) and (35),

$$i_{\tau-1} = \left(1 + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}\right) \frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1}}{R_{\tau-1}^{\frac{1}{\alpha}}}. \tag{36}$$

Similarly as $n_{t-2,1,-j} = \frac{n}{\mu}$ and $n_{t-2,2,-j} = \frac{n}{\mu^{\frac{1}{2}}}$,

$$k_{\tau-1,1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L \left(\frac{n}{\mu}\right)^{\frac{1}{\alpha}}}{R_{\tau-2}^{\frac{1}{\alpha}}}, \tag{37}$$

and

$$k_{\tau-1,2} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L \left(\frac{n}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}}}{R_{\tau-2}^{\frac{1}{\alpha}}}. \tag{38}$$

Combining (5) with (37) and (38),

$$y_{\tau-1} = \left(\left(\frac{1}{\mu}\right)^{\frac{1}{\alpha}-1} + \left(\frac{1}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}-1}\right)(1-\alpha)^{\frac{1}{\alpha}-1} L n^{\frac{1}{\alpha}-1} \frac{1}{R_{\tau-2}^{\frac{1}{\alpha}-1}}. \tag{39}$$

From the market-clearing condition for the consumption good, $y_{\tau-1} = x_{\tau-1} + i_{\tau-1}$, we can pin $x_{\tau-1}$ down:

$$x_{\tau-1} = \left(\frac{\left(\frac{1}{\mu}\right)^{\frac{1}{\alpha}-1} + \left(\frac{1}{\mu^{\frac{1}{2}}}\right)^{\frac{1}{\alpha}-1}}{(1-\alpha)R_{t-2}^{\frac{1}{\alpha}-1}} - \frac{1+\mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}}{R_{t-1}^{\frac{1}{\alpha}}}\right)(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1}. \tag{40}$$

Period τ. Recall that implementation takes place in both sectors now. Technology in the two sectors is $n_{\tau,1,-j} = n < n_{\tau,1,j} = \mu n$ and $n_{\tau,2,-j} = \mu^{\frac{1}{2}} n < n_{\tau,2,j} = \mu^{\frac{3}{2}} n$, which implies that in $\tau+1$ capital will be

$$k_{\tau+1,1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}}}{R_{\tau}^{\frac{1}{\alpha}}}, \tag{41}$$

and

$$k_{\tau+1,2} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L (\mu^{\frac{1}{2}} n)^{\frac{1}{\alpha}}}{R_{\tau}^{\frac{1}{\alpha}}}. \tag{42}$$

Then as $i_\tau = \frac{k_{\tau+1,1}}{\mu n} + \frac{k_{\tau+1,2}}{\mu^{\frac{3}{2}} n}$,

$$i_\tau = \left(\frac{1+\mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}}{\mu}\right)\frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1}}{R_t^{\frac{1}{\alpha}}}. \tag{43}$$

Combining (5) with (34) and (35),

$$y_\tau = \left(1+\mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}\right)(1-\alpha)^{\frac{1}{\alpha}-1} L n^{\frac{1}{\alpha}-1} \frac{1}{R_{t-1}^{\frac{1}{\alpha}-1}}. \tag{44}$$

Market-clearing then implies

$$x_\tau = \left(1 + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}\right)\left(\frac{1}{(1-\alpha)R_{\tau-1}^{\frac{1}{\alpha}-1}} - \frac{1}{\mu R_\tau^{\frac{1}{\alpha}}}\right)(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1}. \tag{45}$$

Period τ+1. Recall that like in $\tau-1$ an idea reaches a firm in sector 1, but it is saved for $\tau+2$, when the next boom takes place, and, effectively it is as if it had never arrived. Technology in the two sectors is $n_{\tau+1,1,-j} = n_{\tau+1,1,j} = \mu n$ and $n_{\tau+1,2,-j} = n_{\tau+1,2,j} = \mu^{\frac{3}{2}} n$, which implies capital in $\tau+2$ will be

$$k_{\tau+2,1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L (\mu n)^{\frac{1}{\alpha}}}{R_{\tau+1}^{\frac{1}{\alpha}}}, \tag{46}$$

and

$$k_{\tau+2,2} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L \mu^{\frac{1}{\alpha}} (\mu^{\frac{3}{2}} n)^{\frac{1}{\alpha}}}{R_{\tau+1}^{\frac{1}{\alpha}}}. \tag{47}$$

Then as $i_{\tau+1} = \frac{k_{\tau+2,1}}{\mu n} + \frac{k_{\tau+2,2}}{\mu^{\frac{3}{2}} n}$,

$$i_{\tau+1} = \mu^{\frac{1}{\alpha}-1}\left(1 + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}\right)\frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1}}{R_{\tau+1}^{\frac{1}{\alpha}}}. \tag{48}$$

Combining (5) with (41) and (42),

$$y_{\tau+1} = \left(1 + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}\right)(1-\alpha)^{\frac{1}{\alpha}-1} L n^{\frac{1}{\alpha}-1} \frac{1}{R_\tau^{\frac{1}{\alpha}-1}}. \tag{49}$$

Market-clearing then implies

$$x_{\tau+1} = \left(1 + \mu^{\frac{1}{2}(\frac{1}{\alpha}-1)}\right)\left(\frac{1}{(1-\alpha)R_\tau^{\frac{1}{\alpha}-1}} - \frac{\mu^{\frac{1}{\alpha}-1}}{R_{\tau+1}^{\frac{1}{\alpha}}}\right)(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1}. \tag{50}$$

As $\alpha \in (\frac{1}{2}, 1)$, we can check that $x_t > 0$ for each t.

3.2.2 Equilibrium.

An implementation cycle requires the firm in sector 1 receiving a patent in an odd period, say $\tau-1$, to find it optimal to wait exactly for *one* period before implementing, and the firm in sector 2 receiving a patent in the following even period τ to find it optimal to implement immediately.

For the former to happen the firm's in sector 1 discounted $\tau+1$ profits in the cyclical equilibrium (recall profits are realized one period after implementation) must exceed its discounted profits in τ, which would be realized were it to implement immediately; these are from the viewpoint of period $\tau-1$, which is the period the decision is made. That is

$$\frac{\Pi^{\tau-1}_{\tau+1,1}}{R_{\tau-1}R_\tau} > \frac{\Pi^{\tau-1}_{\tau,1}}{R_{\tau-1}},$$

which is equivalent to

$$\frac{1}{R_\tau} > \left(\frac{R_\tau}{R_{\tau-1}}\right)^{\frac{1}{\alpha}-1}. \tag{51}$$

For the latter to happen, we need (see (15) above)

$$\frac{\Pi^{\tau}_{\tau+1,i}}{R_\tau} \geq \frac{\Pi^{\tau}_{\tau+2,i}}{R_\tau R_{\tau+1}},$$

which is equivalent to

$$\left(\frac{R_{\tau+1}}{R_\tau}\right)^{\frac{1}{\alpha}-1} \geq \frac{1}{R_{\tau+1}}. \tag{52}$$

For a synchronized implementation (*cyclical*) equilibrium to be sustained, none must find it optimal to postpone implementation to some period after the 2-period cycle. For this to happen, it suffices to show that the firm in sector 1 must not wait for the next cycle, i.e. period $\tau+2$. Using the same argument as in Shleifer (1986), and, given the stationary structure of our economy, if a firm postpones implementation from $\tau-1$ to τ, i.e. when (51), then it will also find it optimal to postpone it from $\tau+1$ to $\tau+2$; in other words, (51) shows implementation can only take place in even periods. Thus, showing that a firm prefers implementing in τ to implementing in $\tau+2$ or, by the same token in any future even period, is what we need for our analysis to be complete. Notice that we have made no use of the fact that a new patent will reach the sector in question in $\tau+1$, rendering the $\tau-1$ patent obsolete. It must be then that,

$$\frac{\Pi^{\tau-1}_{\tau+1,i}}{R_{\tau-1}R_\tau} > \frac{\Pi^{\tau-1}_{\tau+3,i}}{R_{\tau-1}R_\tau R_{\tau+1}R_{\tau+2}},$$

which given that $R_{\tau-1} = R_{\tau+1}$ and $R_\tau = R_{\tau+2}$ simplifies to

$$R_\tau R_{\tau+1} > 1. \tag{53}$$

Condition (53) implies that profits are discounted at an on average positive (net) interest rate, and may be viewed as a transversality-like condition. Check that (51) and (53) imply (52), and are necessary conditions for a cyclical equilibrium.

Claim 1 (No-Storage)**.** $R_t \geq 1$ *for each* t *is a necessary and sufficient condition for a no-storage equilibrium.*

Proof. See Appendix A. □

From (16), (18), (40) and (50), we get that

$$\lambda = \mu^{\frac{1}{\alpha}-1} > 1, \text{as } \mu > 1. \tag{54}$$

Lemma 1. *Remark 1 and (54) imply (53).*

Combining (18) with (45) and (50) and using the fact that $R_{\tau-1} = R_{\tau+1}$,

$$\frac{1}{(1-\alpha)R_{\tau}^{\frac{1}{\alpha}-1}} - \frac{\mu^{\frac{1}{\alpha}-1}}{R_{\tau+1}^{\frac{1}{\alpha}}} = v\left(\frac{1}{(1-\alpha)R_{\tau+1}^{\frac{1}{\alpha}-1}} - \frac{1}{\mu R_{\tau}^{\frac{1}{\alpha}}}\right). \tag{55}$$

Rearranging yields

$$\frac{R_\tau}{1-\alpha} + \frac{v}{\mu} = \left(\frac{R_\tau}{R_{\tau+1}}\right)^{\frac{1}{\alpha}}\left(\frac{vR_{\tau+1}}{1-\alpha} + \mu^{\frac{1}{\alpha}-1}\right)$$

From (18) we get R_τ and $R_{\tau+1}$, and as λ is given by (54), then this is equivalent to

$$\frac{v^\gamma}{\rho(1-\alpha)} + \frac{v}{\mu} = \left(\frac{v^{2\gamma}}{\mu^{\gamma(\frac{1}{\alpha}-1)}}\right)^{\frac{1}{\alpha}}\left(\frac{v\mu^{\gamma(\frac{1}{\alpha}-1)}}{\rho v^\gamma(1-\alpha)} + \mu^{\frac{1}{\alpha}-1}\right).^{9} \tag{56}$$

Then, the necessary condition for an implementation cycle (51), combined with (54), and for v given by (56), becomes

$$\rho > \left(\frac{v^{2\gamma}}{\mu^{\frac{1}{\alpha}-1}}\right)^{\frac{1}{\alpha}-1} v^\gamma. \tag{57}$$

Claim 2. *It is* $R_\tau^S < R^I < R_{\tau+1}^S$.

Proof. See Appendix A. □

Claim 2 offers another way to realize the effect of strategic complementarities: as $R_\tau^S < R^I$, then by (14) profits of the follower (a firm in sector 2) are higher in the synchronized rather than the immediate implementation equilibrium.

Claims 1 and 2 together imply that a low v, but, on the other hand makes (57) more likely to hold, on the other hand, makes no-storage less likely to hold.

[9] Parameter v is independent of the economy's starting point in period 1, and, as a result, so are the equilibrium interest rates and the decision to implement. A proof is available on request.

Transversality Condition. The transversality condition is equivalent to requiring the present value of the stream of profits of both firms to be finite. It can be shown (see Appendix A) that for both equilibria in this economy, this boils down to the following condition:

$$\mu^{\frac{1}{\alpha}-1} < \frac{\mu^{\gamma(\frac{1}{\alpha}-1)}}{\rho^2}. \tag{58}$$

The transversality condition is more easily satisfied the higher the concavity (γ) of the utility function is. Bearing in mind that $\mu \geq 1$, the transversality condition will be always met for $\gamma \geq 1$, whereas for $\gamma < 1$, it will be surely violated for ρ close to 1. However, for a sufficiently low γ and a sufficiently high μ, the transversality condition will be violated. As will become clear from the comparative statics below, a low γ will generally imply a low interest rate at the period of the implementation boom, which is the period preceding the one profits are made; this, combined with a sufficiently high innovation rate (μ) can result in non-converging profits for the monopolists. In other words, the higher the innovation rate, μ, is, the lower the curvature of the utility function, γ, is, and the greater the discount factor, ρ, is the more difficult it is for the transversality condition to be satisfied.

The following proposition can then be generated:

Proposition 1. *For a 2-period implementation cycle with capital to be generated as a perfect-foresight equilibrium, (57), Claim 1, and the transversality condition (58) are necessary and sufficient conditions.*

Below attention will be restricted to the case where $\alpha = \frac{2}{3}$. Before generalizing, I proceed with two illustrative cases: the logarithmic preferences case, which generates analytical solutions, and, the case where $\gamma = \frac{1}{2}$, for which results are based on numerical approximations.

Case 1: $\gamma = 1$ (logarithmic preferences). For the competitive price and quantity to be set, (13) requires $\mu \leq 3$. After plugging the parameter values in (56), we get

$$\left(\frac{3}{\rho} + \frac{1}{\mu}\right) v = \frac{v^3}{\mu^{\frac{1}{4}}} \left(\frac{3}{\rho} + 1\right). \tag{59}$$

For $v > 0$, rearranging (59) yields

$$v = \left(\frac{3\mu + \rho}{\mu^{\frac{3}{4}}(3 + \rho)}\right)^{\frac{1}{2}}. \tag{60}$$

Remark 3. *For $\gamma = 1$, it is $1 < v < \mu^{\frac{1}{8}}$.*

Proof. For the left inequality, notice that v is increasing in μ and decreasing in ρ. Then, for $\mu \to 1$ and $\rho \to 1$, $v \to \inf v = 1$. For the right inequality, check that $v < \mu^{\frac{1}{8}}$ implies $\mu > 1$, which is true. Indeed, one can check that for $\gamma = 1$, $v < \sup v = 3^{\frac{1}{8}}$. □

Interest rates (see also (18)) are

$$R_t = \begin{cases} \frac{v}{\rho}, & \text{if } t \text{ even} \\ \frac{\mu^{\frac{1}{2}}}{v\rho}, & \text{if } t \text{ odd} \end{cases}, \tag{61}$$

where $\lambda = \mu^{\frac{1}{2}}$ is given by (54) and v is given by (60). Remark 3 implies net real interest rates are non-negative, thereby rendering storage unprofitable. For $\gamma = 1$, (57) becomes [10]

$$\rho > \frac{v^2}{\mu^{\frac{1}{4}}} = \left(\frac{v}{\mu^{\frac{1}{8}}}\right)^2. \tag{62}$$

The following remark can be generated:

Remark 4. *For logarithmic preferences we can see: (i) From (60) and (61) that v, and R_τ and $R_{\tau+1}$, increase as the innovation rate (μ) increases and the discount factor (ρ) decreases; (ii) from (62) that, the higher the discount factor (ρ) and, the innovation growth rate (μ) are. In the limit as $\rho \to 1$, Remark 3 implies an implementation cycle can be always generated* [11].

For $\gamma = 1$, the transversality condition (58) is always met.

Case 2: $\gamma = \frac{1}{2}$. Given the parameters (56) becomes

$$\frac{3v^{\frac{1}{2}}}{\rho} + \frac{v}{\mu} = \frac{3v^2}{\rho\mu^{\frac{1}{8}}} + \mu^{\frac{1}{8}} v^{\frac{3}{2}}. \tag{63}$$

Then, as by (54) $\lambda = \mu^{\frac{1}{2}}$, (18) implies interest rates are

$$R_t = \begin{cases} \frac{v^{\frac{1}{2}}}{\rho}, & \text{if } t \text{ even} \\ \left(\frac{\mu^{\frac{1}{2}}}{v}\right)^{\frac{1}{2}} \frac{1}{\rho}, & \text{if } t \text{ odd} \end{cases}, \tag{64}$$

where v is given by (63) . Condition (51) becomes

$$\rho > \frac{v}{\mu^{\frac{1}{8}}}. \tag{65}$$

[10] For example, we can set $\mu = 3$, $\rho = 0.99$; then from (60), $v \simeq 1.048$, while from (61) we get $R_\tau = 1.059$ and $R_{\tau+1} = 1.669$. Then in (62) $LHS \simeq 0.99$ and $RHS \simeq 0.835$, and, as a result, the synchronized implementation equilibrium is sustained. An implementation cycle can be generated by setting $\rho = 0.95$ and $\mu = 1.5$. No storage takes place in either equilibrium.

[11] As opposed to here, in Shleifer (1986) a cycle cannot be generated for $\gamma \geq 1$.

Numerical example: Setting $\mu = 1.25$ and $\rho = 0.95$ in (64), yields $R_\tau \simeq 1.038$ and $R_{\tau+1} \simeq 1.128$. Then in (51), $LHS \simeq 0.9632$ and $RHS \simeq 0.95917$, and the synchronized implementation equilibrium is sustained. The transversality condition (58) in this example is satisfied, whereas the positive *net* real interest rates rule out storage in equilibrium.

Based on numerical simulations, the following remark can be generated:

Remark 5. *For $\gamma = \frac{1}{2}$ (i) v and, consequently, R_τ, increase as the discount factor (ρ) decreases, whereas, for high enough values of the discount factor, ρ, they decrease in the innovation rate (μ), and increase, otherwise. $R_{\tau+1}$ increases as the innovation rate (μ) increases and the discount factor (ρ) decreases; (ii) from (65) that, the higher the discount factor (ρ) and the innovation rate (μ) are, the more easily implementation cycles are sustained.*

For $\gamma = \frac{1}{2}$, $v(\rho \to 1, \mu = 3) = 0.91 < v < v(\rho \to 1, \mu = 1) = 1$. For $\rho \to 1$, storage can prevail in equilibrium, let aside the fact that the transversality condition[12] will be surely violated in the limit.

Comparative Statics. Numerical approximations reveal that for fixed innovation rate, μ, and discount factor, ρ, v, which shows the consumption allocation smoothing prevailing in equilibrium, is increasing in the concavity of the utility function, γ. Then, (18) implies R_τ becomes higher and $R_{\tau+1}$ lower. This happens because a high curvature of the utility function reveals a pronounced desire of the representative agent to borrow against his future profits and, thereby smooth his stream of consumption. For the agent not to act in such a way, a high interest rate must prevail in the period preceding the one profits are made and his consumption is boosted (R_τ in our case), so that the agent is deterred from borrowing. But, then (51) implies an implementation cycle is less likely to be sustained as an equilibrium. Then, we can conclude that a high curvature of the utility function renders cycles less likely to be sustained. On the other hand, a very low curvature, or equivalently a very high intertemporal elasticity of substitution ($\frac{1}{\gamma}$), implying according to the above a very low v, can result in storage, or/and violate the transversality condition.

For fixed concavity of the utility function, γ, and, innovation rate, μ, the greater the discount factor ρ is, the lower v and the interest rates are. We can see from (57), that a high ρ implies a more easily sustained cycle: the more patient the agent is, the more likely he is to wait till the period his consumption is boosted, without borrowing.

For fixed curvature and time-preference, the role of the innovation rate, μ, in determining the allocation of consumption between periods, v, actually depends on these. For concavity lower but sufficiently close to 1 or greater than it, v, as well as the interest rates, is increasing in μ (see also

[12] From (58), transversality condition for $\gamma = \frac{1}{2}$ requires $\mu^{\frac{1}{4}} < \frac{1}{\rho^2}$.

Remark 4), and cycles more easily prevail in equilibrium; the greater μ is the greater profits for the capital-good firms will be, and the greater the consumption of the agent will be in the period following the implementation boom. However, for preferences with low γ's (see also Remark 5), for a high time preference, v is decreasing in μ, and, as as result R_τ decreases, whereas $R_{\tau+1}$ increases. These imply an even more likely to be sustained cyclical equilibrium. For a low γ and low ρ, v is increasing in μ.

The following Corollary summarizes the results.

Corollary 2. *The sufficiently lower the concavity (γ) of the agent's utility function is, the more he values future (ρ) and, given his sufficiently high valuation of the future, the greater the innovation rate (μ) is, the more easily implementation cycles are sustained.*

However, the greater these parameters' changes become, the more difficult it is for the transversality condition to be satisfied, and similarly, for the no-storage condition. Therefore, for an equilibrium to be sustained the parameters must balance these opposite effects.

Balanced Growth Path. As $\lambda = \mu^{\frac{1}{\alpha}-1}$ by (54), and $R_{\tau-1} = R_{\tau+1}$ by (18), we can check from (39), (49), and (36), (48), respectively, that along the balanced growth path

$$\frac{y_{\tau+1}}{y_{\tau-1}} = \frac{i_{\tau+1}}{i_{\tau-1}} = \frac{x_{\tau+1}}{x_{\tau-1}} = \mu^{\frac{1}{\alpha}-1}. \tag{66}$$

To study the dynamics of output and investment, from (44), (49), and (43), (48), respectively, we get

$$\frac{y_{\tau+1}}{y_\tau} = \left(\frac{R_{\tau+1}}{R_\tau}\right)^{\frac{1}{\alpha}-1}$$

and

$$\frac{i_{\tau+1}}{i_\tau} = \left(\frac{\mu R_\tau}{R_{\tau+1}}\right)^{\frac{1}{\alpha}}.$$

As interest rates are given by (18), these become

$$\frac{y_{\tau+1}}{y_\tau} = \left(\frac{\mu^{\frac{1}{\alpha}-1}}{v^2}\right)^{\gamma(\frac{1}{\alpha}-1)} \tag{67}$$

and

$$\frac{i_{\tau+1}}{i_\tau} = \mu^{\frac{1}{\alpha}}\left(\frac{v^2}{\mu^{\frac{1}{\alpha}-1}}\right)^{\frac{\gamma}{\alpha}}. \tag{68}$$

We can now revisit the above cases.

Case 1: $\gamma = 1$ (logarithmic preferences) (cont'd). (67) and (68) respectively become

$$\frac{y_{\tau+1}}{y_\tau} = \frac{\mu^{\frac{1}{4}}}{v} \tag{69}$$

and

$$\frac{i_{\tau+1}}{i_\tau} = \mu^{\frac{3}{4}} v^3. \tag{70}$$

It follows from Remark 3 that $\frac{y_{\tau+1}}{y_\tau} \geq \frac{x_{\tau+1}}{x_\tau}$, implying consumption is more volatile than output. Remark 3, in combination with (69), further implies $1 \leq \frac{y_{\tau+1}}{y_\tau} \leq \mu^{\frac{1}{8}}$.

Remark 3 and (70) imply $\frac{i_\tau}{i_{\tau-1}} \leq 1$ and $\frac{i_{\tau+1}}{i_\tau} \geq \mu^{\frac{1}{2}}$: in the implementation period, τ, investment is lower than in $\tau - 1$, whereas it grows proportionally by considerably more than its trend in the period profits are realized. In other words investment exhibits 'undershooting', although in the period of the implementation boom, interest rates are lower. To see why this happens, recall that firms implementing their new patents, produce, after controlling for the interest rates, the same amount of capital as before, but at a lower cost. This requires less of the consumption good to be saved, or, which is the same, less investment. As for capital, $\frac{k_{t+1,i}}{k_{t-1,i}} = \mu^{\frac{3}{2}}$, whereas $\frac{k_{t+1,i}}{k_{t,i}} = \left(\frac{\mu^{\frac{1}{4}}}{v}\right)^3$. Observe that capital, naturally, has a pattern similar to the one of output. The above are collected in the following remark:

Remark 6. *For logarithmic preferences, and after controlling for departures from trend, we can see that: (i) Consumption is procyclical, whereas investment countercyclical; (ii) investment is more volatile than consumption which is more volatile than output; (iii) investment 'undershoots'.*

Case 2 (cont'd): $\gamma = \frac{1}{2}$. (67) and (68) respectively become

$$\frac{y_{\tau+1}}{y_\tau} = \left(\frac{\mu^{\frac{1}{4}}}{v}\right)^{\frac{1}{2}} \tag{71}$$

and

$$\frac{i_{\tau+1}}{i_\tau} = \mu^{\frac{9}{8}} v^{\frac{3}{2}}. \tag{72}$$

Above we showed that $0.91 < v < 1$. Then,

$$\frac{x_{\tau+1}}{x_\tau} < 1 \ll \lambda^{\frac{1}{2}} = \mu^{\frac{1}{4}}.$$

As for the patterns of output and investment, (see (71) and (72) above), this respectively implies, that $1 < \frac{y_{\tau+1}}{y_\tau} < 1.2$ and $\frac{i_{\tau+1}}{i_\tau} > \mu^{\frac{1}{2}}$, with the latter implying $\frac{i_\tau}{i_{\tau-1}} < 1$. The following remark can be generated:

Remark 7. *For $\gamma = \frac{1}{2}$, and after controlling for departures from trend, we can see that: (i) Consumption is procyclical, whereas investment countercyclical; (ii) investment is more volatile than consumption which is more volatile than output; consumption overshoots ($v \leq 1$).*

Along the lines of Remarks 6 and 7, and based on numerical simulations, the following corollary can be generated:

Corollary 3. *In the synchronized implementation equilibrium and, after controlling for departures from trend: (i) Consumption is procyclical, whereas investment is countercyclical; (ii) investment is more volatile than output and consumption; (iii) for sufficiently high concavity (γ) and sufficiently low innovation rate (μ), consumption is more volatile than output; (iv) for $\rho \to 1$ and γ close enough to 1 at its minimum, as the concavity (γ) of the utility function increases: a) consumption becomes less volatile, b) investment and output becomes less volatile when μ is high, and more volatile otherwise, and (v) investment (almost always) undershoots, while consumption overshoots for low enough γ.*

4 Welfare Analysis (in progress)

No matter which equilibrium the agents' expectations support, it will, inherently, be Pareto suboptimal. This is due to the monopoly profits assumed here. The analysis below attempts to Pareto rank the equilibria. Before generalizing, the focus will be on the cases presented previously.

4.1 Case 1: $\gamma = 1$ (logarithmic preferences).

In section 2, each equilibrium case was analyzed independently of the other. That meant that the economy may well have started with different levels of capital in the two equilibria. It was claimed, subsequently, that an immediate implementation equilibrium is always possible, whereas for appropriate values of the parameters, $\{\mu, \rho, \gamma\}$, a synchronized implementation equilibrium could be generated. A direct implication of this is that the two cases are not directly comparable. If an economy starts in one equilibrium with a higher level of capital than the other one, it is likely to generate higher consumption, and hence, greater utility for the representative agent as well.

The following thought process will be followed: I will continue treating the two equilibria separately, providing each economy with the appropriate initial level of capital-goods. Subsequently, the present values of the two utility streams will be compared. I will show that, an acyclical equilibrium generates higher welfare than the cyclical one, although in the former a lower initial level of investment is required.

4.1.1 Immediate Implementation

Utility. Recall from section 2 that in the immediate implementation equilibrium, growth is perfectly smooth:

$$\frac{x_{\tau+1}}{x_\tau} = \frac{x_\tau}{x_{\tau-1}} = \mu^{\frac{1}{4}}.$$

Then, the present value, from the viewpoint of period 1, of the stream of consumption in periods $1, \ldots, \infty$, is

$$U_I = \log(x_1^I) + \rho \log(\mu^{\frac{1}{4}} x_1^I) + \rho^2 \log(\mu^{\frac{1}{2}} x_1^I) + \rho^3 \log(\mu^{\frac{3}{4}} x_1^I) + \ldots \ ,$$

which is equivalent to

$$U_I = \frac{\log(x_1^I)}{1-\rho} + \frac{\rho}{(1-\rho)^2} \log(\mu^{\frac{1}{4}}). \tag{73}$$

x_1^I denotes consumption in period 1; from (24) and (32), and for $n = 1$, this is equal to

$$x_1^I = \left(\frac{1}{3}\right)^{\frac{3}{2}} L \frac{\rho^{\frac{1}{2}}}{\mu^{\frac{1}{8}}} \left[3\left(1 + \frac{1}{\mu^{\frac{1}{4}}}\right) - \rho\left(1 + \frac{1}{\mu^{\frac{5}{4}}}\right)\right]. \tag{74}$$

Initial Level of Output. From (10), and, (22) and (23), the initial amount of final good (investment) the economy needs to start with is

$$i_0^I = \frac{\left(\frac{1}{3}\right)^{\frac{3}{2}} L}{R^{I\frac{3}{2}}} \left(\frac{1}{\mu} + \frac{1}{\mu^{\frac{5}{4}}}\right), \tag{75}$$

where $R^I = \frac{\mu^{\frac{1}{4}}}{\rho}$.

4.1.2 Synchronized Implementation

Utility. In the synchronized implementation equilibrium, the present discounted value of lifetime utility is

$$U_S = \log(x_1^S) + \rho \log\left[\left(\frac{\mu^{\frac{1}{2}}}{v}\right) x_1^S\right] + \rho^2 \log(\mu^{\frac{1}{2}} x_1^S) + \rho^3 \log\left[\left(\frac{\mu^{\frac{1}{2}}}{v}\right) \mu^{\frac{1}{2}} x_1^S\right] + \ldots \ .$$

This is equivalent to

$$U_S = \frac{\log(x_1^S)}{1-\rho} + \frac{\rho(1+\rho)}{(1-\rho^2)^2} \log(\mu^{\frac{1}{2}}) - \frac{\rho}{1-\rho^2} \log v \tag{76}$$

where v is given by (60). From (40) and (18), and for $n = 1$,

$$x_1^S = \left(\frac{1}{3}\right)^{\frac{3}{2}} L \frac{\rho^{\frac{1}{2}}}{\mu^{\frac{1}{4}} v^{\frac{1}{2}}} \left[3\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right) - \rho\frac{v^2}{\mu^{\frac{1}{4}}}\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right)\right]. \tag{77}$$

Initial Level of Output. From (10), and, (37) and (38), the initial amount of final good (investment) the economy needs to start with is

$$i_0^S = \frac{\left(\frac{1}{3}\right)^{\frac{3}{2}} L}{R_\tau^{S\frac{3}{2}}} \left(\frac{1}{\mu^{\frac{3}{2}}} + \frac{1}{\mu^{\frac{5}{4}}}\right), \tag{78}$$

where R_τ^S is given by (61).

Remark 8. *For logarithmic preferences, as $\rho \to 1$, immediate implementation Pareto dominates the synchronized one.*

Proof. See Appendix A. □

5 Balanced Growth Path with N Capital-Good Sectors (in progress)

Suppose there are $N > 2$ capital-good sectors in the economy, with a sufficiently high number of firms in each one. The production of the final-good firm is now given by

$$F_t(L, \{k_{t,i}\}_{i=1}^N) = L_t^\alpha \sum_{i=1}^N k_{t,i}^{1-\alpha}. \tag{79}$$

The rest of the assumptions remains unchanged.[13]

Time and Symmetry. At $t-N+1$, where $t \geq N$, an idea arrives to the first sector, at $t-N+2$ an idea arrives to the second sector, and so on for the remaining $N-2$ sectors. Then, by t, all sectors will have received an idea related to the capital-good they specialize. To impose symmetry across sectors with respect to the leading advantage in state of the art technology, the $t-N$ technology in sector 1 is n, in sector 2 it is $\mu^{\frac{1}{N}} n$, in sector 3 $\mu^{\frac{2}{N}} n$, ..., in sector N it is $\mu^{\frac{N-1}{N}} n$.

Stationarity. Stationarity requires that interest rates are repeated every N periods. Consequently, the following restrictions will be imposed along the balanced growth path:

$$\frac{x_{t+1}}{x_{t-N+1}} = \lambda \tag{80}$$

and

$$\frac{x_{t-N+j+1}}{x_{t-N+j}} = v_j > 0, \text{where j=1, ...,N.} \tag{81}$$

[13]All derivations throughout Section 5 are collected in Appendix B.

Combined with the first order conditions from the agent's utility maximization problem (like in (3) and (4)), these imply

$$R_{t-N+j} = \frac{v_j^\gamma}{\rho}, \text{where j=1, \ldots ,N.} \tag{82}$$

Remark 9. *Check that* $R_{t-N+1} \cdot \dots \cdot R_t = \frac{\lambda^\gamma}{\rho^N}$.

5.1 Immediate Implementation

Steady-Growth Equilibrium. For interest rate $R = \frac{\mu^{\frac{\gamma}{N}(\frac{1}{\alpha}-1)}}{\rho}$ prevailing in each period,

$$\frac{y_{\tau+1}}{y_\tau} = \frac{y_\tau}{y_{\tau-1}} = \frac{i_{\tau+1}}{i_\tau} = \frac{i_\tau}{i_{\tau-1}} = \frac{x_{\tau+1}}{x_\tau} = \frac{x_\tau}{x_{\tau-1}} = \mu^{\frac{1}{N}(\frac{1}{\alpha}-1)}. \tag{83}$$

As the same positive net interest rate prevails every period, no firm has an incentive to postpone implementation past the period it receives the idea. Hence, in line with Corollary 1, an acyclical (steady-growth) perfect-foresight equilibrium is always attainable.

5.2 Synchronized Equilibrium

With N capital-good sectors, any equilibrium with periodicity $\leq N$ may prevail, when the commonly shared expectations are accordingly shaped. I will first characterize the (symmetric) N-sector, N-period equilibrium, and afterwards the (asymmetric) N-sector, $< N$-period equilibrium.

5.2.1 N-sector, N-period Equilibrium

Based on the equilibrium conditions, (80) and (81), I will restrict attention to the the following balanced growth path equilibrium:

$$\frac{x_{t-N+2}}{x_{t-N+1}} = v_1\,, \frac{x_{t-N+j+1}}{x_{t-N+j}} = 1\,, \frac{x_t}{x_{t-1}} = v_{N-1}\,, \frac{x_{t+1}}{x_t} = v_N\,, \tag{84}$$

where $j = 2, \ldots, N-2$.

Hence, the N-periodic interest rates to prevail in equilibrium are

$$R_1 = \frac{v_1^\gamma}{\rho}\,, R_j = \frac{1}{\rho}\,, \text{where j=2, \ldots, N-2}\,, R_{N-1} = \frac{v_{N-1}^\gamma}{\rho}\,, R_N = \frac{v_N^\gamma}{\rho}\,. \tag{85}$$

Like in Section 3, $\lambda = \mu^{\frac{1}{\alpha}-1}$. This combined with Remark 9 and (85) implies $v_1 \times v_{N-1} \times v_N = \mu^{\frac{1}{\alpha}-1}$.

Furthermore, Remark 9 for $\lambda = \mu^{\frac{1}{\alpha}-1}$ implies $R_{t-N+1} \cdot \dots \cdot R_t > 1$. Then, the transversality-like condition (see also (53)) preventing a firm from waiting past the cycle, without accounting for the fact that another idea will outdate the one under its belt, will be met. Hence, the following condition will be necessary and sufficient for a perfect-foresight N-period cyclical equilibrium to be generated (see also Lemma 1 and the analysis in subsection 3.2.2), under the premise that the transversality and the no-storage conditions are satisfied:

$$\frac{\Pi_{t+1,1}}{R_{t-N+1} \cdot \dots \cdot R_t} > \frac{\Pi_{t-N+2,1}}{R_{t-N+1}},$$

which boils down to

$$\frac{1}{R_{t-N+2} \cdot \dots \cdot R_t} > \left(\frac{R_t}{R_{t-N+1}}\right)^{\frac{1}{\alpha}-1}. \tag{86}$$

Combined with (85) and, after rearranging, this yields

$$\rho^{N-1} > \left(\frac{\mu v_N}{v_1}\right)^{\gamma(\frac{1}{\alpha}-1)} \frac{1}{v_1^{\gamma}}. \tag{87}$$

6 Extensions

In progress are still the welfare analysis and the N-sector case which will complete the paper. Calibrating and comparing the theoretical results developed here with data and other empirical work lies within the spectrum of the analysis. Future work could include introducing uncertainty and studying fiscal policy, in both complete and incomplete markets.

A Appendix: Proofs

Proof of Claim 1 (No-Storage): There are two kinds of storable commodities in the economy: the capital-goods and the final-good. I will deal with each one separately.

Capital-goods. The capital-goods can be either stored by the capital-good firms, or the final-good firm. Both cases will be checked in that order.

Capital-good firms. Suppose capital-good firm i produces an additional unit of capital-good in $t-1$, which is available as a production input in t, but is instead stored and sold to the final-good firm in $t+1$. The cost of producing it, as of $t-1$, is $\frac{1}{n_{t-1,i,-j}}$, whereas the value of revenue generated, as of $t-1$, is $\frac{q_{t+1}}{R_t R_{t-1}}$. As q is given by (11), no storage will take place if

$$\frac{R_t}{n_{t-1,i,-j}R_tR_{t-1}} \leq \frac{1}{n_{t-1,i,-j}},$$

which is equivalent to

$$R_{t-1} \geq 1. \tag{88}$$

Therefore, positive net interest rates imply no-storage taking place in equilibrium.

If the capital-firm decides to sell the additional unit of capital in $t+1$, then, after controlling for the fact that a new idea may render the old one obsolete, the no-storage condition becomes

$$R_tR_{t-1} \geq 1,$$

which is satisfied (see Lemma 1). This further implies that beyond 2 periods (88) is sufficient.

For the possibility of a new patent rendering the existing one obsolete, (88) is sufficient but not necessary, as revenue is lower, the lower the cost of production is. This completes the proof for the capital-good. The final-good can be also stored by the final-good firms.

Final-good firms. Suppose the final-good firm buys (rents) a unit of capital type 1 in τ and sells it in $\tau+1$. For it not to happen, it must be

$$q_{\tau,1} > \frac{q_{\tau+1,1}}{R_\tau}.$$

From (11), this is equivalent to

$$\frac{R_{\tau-1}}{n} > \frac{R_\tau}{R_\tau n},$$

which implies $R_{\tau+1} > 1$.

Respectively, from $\tau+1$ to $\tau+2$, condition $q_{\tau+1,1} > \frac{q_{\tau+2,1}}{R_{\tau+1}}$, yields $R_\tau > \frac{1}{\mu}$, as $q_{t+2,1} = \frac{R_{\tau+1}}{\mu n}$. The results are the same for capital-good 2.

Final-good. The same condition is trivially necessary and sufficient for the final-good. To see this point, recall that all production factors and the interest rates are in terms of it.

Proof of Claim 2: In the immediate implementation balanced growth path equilibrium the equilibrium interest rate prevailing in every period is (see also (32))

$$R = \frac{\mu^{\frac{\gamma}{2}(\frac{1}{\alpha}-1)}}{\rho},$$

while in the synchronized equilibrium, the series of interest rates is provided by (18),

$$R_t = \begin{cases} \frac{v^\gamma}{\rho}, & \text{if } t \text{ even} \\ \left(\frac{\lambda}{v}\right)^\gamma \frac{1}{\rho}, & \text{if } t \text{ odd} \end{cases},$$

where $\lambda = \mu^{\frac{1}{\alpha}-1}$ (see also (54)).

Proving the claim boils down to showing that $v < \mu^{\frac{\frac{1}{\alpha}-1}{2}}$. Define

$$f(v;\gamma,\rho,\mu,\alpha) = \frac{v^\gamma}{\rho(1-\alpha)} + \frac{v}{\mu} - \left(\frac{v^{2\gamma}}{\mu^{\gamma(\frac{1}{\alpha}-1)}}\right)^{\frac{1}{\alpha}} \left(\frac{v\mu^{\gamma(\frac{1}{\alpha}-1)}}{\rho v^\gamma(1-\alpha)} + \mu^{\frac{1}{\alpha}-1}\right),$$

where $v > 0$, and the range of parameters defined as previously. From the graph (insert graph) we can see that $\lim_{v\longrightarrow 0} f(v;\gamma,\rho,\mu,\alpha) > 0$ and $\lim_{v\longrightarrow\infty} f(v;\gamma,\rho,\mu,\alpha) < 0$, whereas $f(\cdot,\cdot)$ is concave and continuous. Then, by the Intermediate Value Theorem there must exist a unique v^* (the equilibrium value) such that $f(v^*;\gamma,\rho,\mu,\alpha) = 0$. Consequently, in order to prove the claim, it suffices to show that $f(\cdot,\cdot)$ is negative for $v = \mu^{\frac{\frac{1}{\alpha}-1}{2}}$. Put it otherwise, $\mu^{\frac{\frac{1}{\alpha}-1}{2}}$ is always greater than the equilibrium v^*. For $v = \mu^{\frac{\frac{1}{\alpha}-1}{2}}$, it is $f(\cdot;\gamma,\rho,\mu,\alpha) = \frac{\mu^{\gamma(\frac{1}{\alpha}-1)/2}}{\rho(1-\alpha)} - \frac{\mu^{\gamma(\frac{1}{\alpha}-1)/2+(\frac{1}{\alpha}-1)/2}}{\rho(1-\alpha)} + \mu^{(\frac{1}{\alpha}-12-1)} - \mu^{(\frac{1}{\alpha}-1)}$, and as $\mu > 1$, a negative value is generated. This completes the proof.

Transversality Condition: The focus will be on the synchronized implementation case. The same condition can be easily shown to hold in the immediate implementation case.

The present value from the viewpoint of period 1 of the stream of profits of a firm is

$$PV_{1,i} = \frac{\Pi_{3,i}}{R_1R_2} + \frac{\Pi_{5,i}}{R_1R_2R_3R_4} + \frac{\Pi_{7,i}}{R_1\dots R_6} + \dots, \text{ where } i = 1,2 \ .$$

Profits are given by (14) and, as in the equilibria considered here, $R_1 = R_3 = \cdots = R_{\tau+1}$, and $R_2 = R_4 = \cdots = R_\tau$, this becomes

$$PV_{1,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L_t n_{t-1,i,-j}}{R_\tau^{\frac{1}{\alpha}-1}} \left(\frac{\mu-1}{\mu}\right) \frac{1}{R_\tau R_{\tau+1}} \left[1 + \frac{\mu^{\frac{1}{\alpha}-1}}{R_\tau R_{\tau+1}} + \left(\frac{\mu^{\frac{1}{\alpha}-1}}{R_\tau R_{\tau+1}}\right)^2 + \dots\right].$$

This is finite if

$$\frac{\mu^{\frac{1}{\alpha}-1}}{R_\tau R_{\tau+1}} < 1.$$

As interest rates are given by (18), the resulting equation is (58).

Proof of Remark 8: The proof consists of two parts:

Part 1. From (73) and (76),

$$U_S - U_I = \frac{\log(x_1^S) - \log(x_1^I)}{1+\rho} + \frac{\rho(1+\rho)}{(1-\rho^2)^2}\log(\mu^{\frac{1}{2}}) - \frac{\rho}{1-\rho^2}\log v - \frac{\rho}{(1-\rho)^2}\log(\mu^{\frac{1}{4}}).$$

Combining this with (74) and (77), and simplifying the last three terms results in

$$U_S - U_I = \frac{1}{1-\rho}\log\left\{\frac{3\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right) - \rho\frac{v^2}{\mu^{\frac{1}{4}}}\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right)}{v^{\frac{1}{2}}\mu^{\frac{1}{8}}\left[3\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right) - \rho\left(1+\frac{1}{\mu^{\frac{5}{4}}}\right)\right]}\right\} + \frac{\rho}{1-\rho^2}\log\left(\frac{\mu^{\frac{1}{4}}}{v}\right).$$

Then, $U_I \geq U_S$ implies

$$\log\left\{\frac{v^{\frac{1}{2}}\mu^{\frac{1}{8}}\left[3\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right) - \rho\left(1+\frac{1}{\mu^{\frac{5}{4}}}\right)\right]}{3\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right) - \rho\frac{v^2}{\mu^{\frac{1}{4}}}\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right)}\right\} \geq \frac{\rho}{1+\rho}\log\left(\frac{\mu^{\frac{1}{4}}}{v}\right). \tag{89}$$

For $\rho \to 1$, and after taking out logarithms, (89) becomes:

$$\frac{v^{\frac{1}{2}}\mu^{\frac{1}{8}}\left[3\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right) - \left(1+\frac{1}{\mu^{\frac{5}{4}}}\right)\right]}{3\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right) - \frac{v^2}{\mu^{\frac{1}{4}}}\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right)} \geq \frac{\mu^{\frac{1}{8}}}{v^{\frac{1}{2}}}.$$

Rearranging yields

$$3\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right)(v-1) \geq \left[v\left(1+\frac{1}{\mu^{\frac{5}{4}}}\right) - \frac{v^2}{\mu^{\frac{1}{4}}}\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right)\right],$$

which is equivalent to

$$\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right)\left[3(v-1) \; + \frac{v^2}{\mu^{\frac{1}{4}}}\right] \geq v\left(1+\frac{1}{\mu^{\frac{5}{4}}}\right).$$

Dropping the term $3(v-1)$, which is non-negative as $v \geq 1$ on the LHS, and simplifying yields

$$\left(1+\frac{1}{\mu^{\frac{1}{4}}}\right)\frac{v}{\mu^{\frac{1}{4}}} \geq \left(1+\frac{1}{\mu^{\frac{5}{4}}}\right). \tag{90}$$

As $v \geq 1$, we can ignore it on the LHS of (90), and after simplifying, we end up with

$$1 \geq \mu^{\frac{3}{4}} - \mu^{\frac{1}{2}}. \tag{91}$$

Differentiating the RHS of (91) with respect to μ yields $\frac{3}{4}\ \mu^{-\frac{1}{4}} - \frac{1}{2}\ \mu^{-\frac{1}{2}} > 0$. But recall that throughout it is assumed that $\mu \in [1,3]$. Then, for $\mu = 3$ the RHS is maximized, yielding ~ 0.55, which is strict less than 1. Recall also that for $\rho \to 1$ (see also Remark 6) a cycle can be always generated.

Part 2. For the proof to be complete, it suffices to show that in the synchronized implementation equilibrium, the economy needs a greater level of the consumption good (investment) to start. It must be then that

$$i_0^S \geq i_0^I,$$

where i_0^S and i_0^I are given by (75) and (78), respectively. Substituting for the interest rates in these, and rearranging yields,

$$\left(\frac{1}{\mu^{\frac{3}{2}}}+\frac{1}{\mu^{\frac{5}{4}}}\right) \geq \left(\frac{v}{\mu^{\frac{1}{4}}}\right)^{\frac{3}{2}} \left(\frac{1}{\mu}+\frac{1}{\mu^{\frac{5}{4}}}\right), \tag{92}$$

where v is given by (60). For $\rho \to 1$, $v \to \left(\frac{3\mu+1}{4\mu^{\frac{3}{4}}}\right)^{\frac{1}{2}}$. Consequently, and after further simplifying, (92) becomes

$$\left(\frac{1}{\mu^{\frac{3}{2}}}+\frac{1}{\mu^{\frac{5}{4}}}\right) \geq \left[\frac{(3\mu+1)^{\frac{3}{4}}}{2^{\frac{3}{2}}\mu^{\frac{15}{16}}}\right] \left(\frac{1}{\mu}+\frac{1}{\mu^{\frac{5}{4}}}\right).$$

This is equivalent to

$$4\mu^{\frac{11}{12}} \geq 3\mu+1.$$

Suppose $f(\mu)=4\mu^{\frac{11}{12}}-3\mu-1$. It is easy to see that $f'(\mu)>0$, for $\mu \in [1,3]$. Then, as $f(1)=0$, the proof is complete.

B Balanced Growth Path with N Capital-Good Sectors

B.1 Immediate Implementation

B.1.1 Demands and Supplies

Period $t-N+1$. An innovation arrives to sector 1 and is immediately implemented. From (12) equilibrium capital in period $t-N+2$ is

$$k_{t-N+2,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{\frac{i-1}{N}} n)^{\frac{1}{\alpha}}}{R_{t-N+1}^{\frac{1}{\alpha}}}, \quad \text{where } i=1,\ldots \text{N}.$$

Then, since $i_{t-N+1} = \frac{k_{t-N+2,1}}{\mu n} + \sum_{i=2}^{N} \frac{k_{t-N+2,i}}{\mu^{\frac{i-1}{N}} n}$,

$$i_{t-N+1} = \left(\frac{1}{\mu}+\sum_{i=2}^{N} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)}\right)(1-\alpha)^{\frac{1}{\alpha}} \frac{Ln^{\frac{1}{\alpha}-1}}{R_{t-N+1}^{\frac{1}{\alpha}}}.$$

Similarly,

$$k_{t-N+1,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{\frac{i-1}{N}} n)^{\frac{1}{\alpha}}}{R_{t-N}^{\frac{1}{\alpha}}}, \text{ where } i = 1, \ldots \text{N-1} \text{ and,}$$

$$k_{t-N+1,N} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}}}{R_{t-N}^{\frac{1}{\alpha}}} \left(\mu^{-\frac{1}{N}}\right)^{\frac{1}{\alpha}}.$$

Then (79) combined with the last two equations yields

$$y_{t-N+1} = \left(\sum_{i=1}^{N-1} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)} + \mu^{-\frac{1}{N}(\frac{1}{\alpha}-1)}\right) \frac{(1-\alpha)^{\frac{1}{\alpha}-1} L n^{\frac{1}{\alpha}-1}}{R_{t-N}^{\frac{1}{\alpha}-1}}.$$

The consumption good's market-clearing condition implies

$$x_{t-N+1} = (1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1} \left(\frac{\sum_{i=1}^{N-1} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)} + \mu^{-\frac{1}{N}(\frac{1}{\alpha}-1)}}{(1-\alpha) R_{t-N}^{\frac{1}{\alpha}-1}} - \frac{\frac{1}{\mu} + \sum_{i=2}^{N} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)}}{R_{t-N+1}^{\frac{1}{\alpha}}} \right).$$

It must be that $x_{t-N+1} > 0$.

Period $t-N+2$. An innovation arrives to sector 2 and is immediately implemented. At the same time imitators enter sector 1. Then,

$$k_{t-N+3,1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu n)^{\frac{1}{\alpha}}}{R_{t-N+2}^{\frac{1}{\alpha}}},$$

and

$$k_{t-N+3,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{\frac{i-1}{N}} n)^{\frac{1}{\alpha}}}{R_{t-N+2}^{\frac{1}{\alpha}}}, \quad \text{where } i = 1, \ldots \text{N}.$$

Then, since $i_{t-N+2} = \frac{k_{t-N+3,1}+k_{t-N+3,2}}{\mu n} + \sum_{i=3}^{N} \frac{k_{t-N+3,i}}{n}$, it is

$$i_{t-N+2} = \left(\mu^{\frac{1}{\alpha}-1} + \frac{\mu^{\frac{1}{N}(\frac{1}{\alpha}-1)}}{\mu} + \sum_{i=3}^{N} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)}\right) \frac{(1-\alpha)^{\frac{1}{\alpha}} L n^{\frac{1}{\alpha}-1}}{R_{t-N+2}^{\frac{1}{\alpha}}}.$$

Production will be

$$y_{t-N+2} = \sum_{i=1}^{N} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)} \frac{(1-\alpha)^{\frac{1}{\alpha}-1} L n^{\frac{1}{\alpha}-1}}{R_{t-N+1}^{\frac{1}{\alpha}-1}}.$$

Consumption, which follows from market-clearing, is

$$x_{t-N+2} = (1-\alpha)^{\frac{1}{\alpha}} Ln^{\frac{1}{\alpha}-1} \left(\frac{\sum_{i=1}^{N} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)}}{(1-\alpha) R_{t-N+1}^{\frac{1}{\alpha}-1}} - \frac{\mu^{\frac{1}{\alpha}-1} + \frac{\mu^{\frac{1}{N}}}{\mu} + \sum_{i=3}^{N} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)}}{R_{t-N+2}^{\frac{1}{\alpha}}} \right).$$

It must again be $x_{t-N+2} > 0$.

Period t**.** In the periods up to t there have been implementation is all sectors but N. Sector N implements then. Then, the following values are generated:

$$k_{t+1,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{(1+\frac{i-1}{N})} n)^{\frac{1}{\alpha}}}{R_t^{\frac{1}{\alpha}}}, \text{ where i=1, \ldots, N-1 },$$

and

$$k_{t+1,N} = \frac{(1-\alpha)^{\frac{1}{\alpha}} Ln^{\frac{1}{\alpha}}}{R_t^{\frac{1}{\alpha}}} \left(\mu^{1-\frac{1}{N}}\right)^{\frac{1}{\alpha}}.$$

$$i_t = \left(\sum_{i=1}^{N-1} \mu^{(1+\frac{i-1}{N})(\frac{1}{\alpha}-1)} + \frac{\mu^{(\frac{N-1}{N})(\frac{1}{\alpha}-1)}}{\mu} \right) \frac{(1-\alpha)^{\frac{1}{\alpha}} Ln^{\frac{1}{\alpha}-1}}{R_t^{\frac{1}{\alpha}}}.$$

As

$$k_{t,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{(1+\frac{i-1}{N})} n)^{\frac{1}{\alpha}}}{R_{t-1}^{\frac{1}{\alpha}}}, \quad \text{where } i \neq N, N-1,$$

and

$$k_{t,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{\frac{i-1}{N}} n)^{\frac{1}{\alpha}}}{R_{t-1}^{\frac{1}{\alpha}}}, \quad \text{where } i = N, N+1.$$

$$y_t = \left(\sum_{i=1}^{N-2} \mu^{(1+\frac{i-1}{N})(\frac{1}{\alpha}-1)} + \sum_{i=N-1}^{N} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)} \right) \frac{(1-\alpha)^{\frac{1}{\alpha}-1} Ln^{\frac{1}{\alpha}-1}}{R_{t-1}^{\frac{1}{\alpha}-1}}.$$

$$x_t = (1-\alpha)^{\frac{1}{\alpha}} Ln^{\frac{1}{\alpha}-1} \left(\frac{\sum_{i=1}^{N-2} \mu^{(1+\frac{i-1}{N})(\frac{1}{\alpha}-1)} + \sum_{i=N-1}^{N} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)}}{(1-\alpha) R_{t-1}^{\frac{1}{\alpha}-1}} - \frac{\sum_{i=1}^{N-1} \mu^{(1+\frac{i-1}{N})(\frac{1}{\alpha}-1)} + \frac{\mu^{(\frac{N-1}{N})(\frac{1}{\alpha}-1)}}{\mu}}{R_t^{\frac{1}{\alpha}}} \right).$$

$x_t > 0$.

Period $t+1$. The second round of implementations starts with a firm in sector 1 implementing first. Following the same steps as above:

$$k_{t+2,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{(1+\frac{i-1}{N})} n)^{\frac{1}{\alpha}}}{R_{t+1}^{\frac{1}{\alpha}}}, \quad \text{where } i = 1, \dots \text{N}.$$

Then, since $i_{t+1} = \frac{k_{t+2,1}}{\mu n} + \sum_{i=2}^{N} \frac{k_{t+2,i}}{\mu^{\frac{i-1}{N}} n}$,

$$i_{t+1} = \left(\frac{\mu^{\frac{1}{\alpha}-1}}{\mu} + \sum_{i=2}^{N} \mu^{(1+\frac{i-1}{N})(\frac{1}{\alpha}-1)} \right) (1-\alpha)^{\frac{1}{\alpha}} \frac{Ln^{\frac{1}{\alpha}-1}}{R_{t+1}^{\frac{1}{\alpha}}}.$$

$$y_{t+1} = \left(\sum_{i=1}^{N-1} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)} + \mu^{-\frac{1}{N}(\frac{1}{\alpha}-1)} \right) \frac{(1-\alpha)^{\frac{1}{\alpha}-1} L(\mu n)^{\frac{1}{\alpha}-1}}{R_t^{\frac{1}{\alpha}-1}}.$$

Market-clearing condition implies

$$x_{t+1} = (1-\alpha)^{\frac{1}{\alpha}} L(\mu n)^{\frac{1}{\alpha}-1} \left(\frac{\sum_{i=1}^{N-1} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)} + \mu^{-\frac{1}{N}(\frac{1}{\alpha}-1)}}{(1-\alpha) R_t^{\frac{1}{\alpha}-1}} - \frac{\frac{1}{\mu} + \sum_{i=2}^{N} \mu^{(\frac{i-1}{N})(\frac{1}{\alpha}-1)}}{R_{t+1}^{\frac{1}{\alpha}}} \right).$$

$x_{t+1} > 0$.

B.1.2 Steady-Growth Equilibrium

If the interest rate $R = \frac{\mu^{\frac{\gamma}{N}(\frac{1}{\alpha}-1)}}{\rho}$ is set in each period,

$$\frac{y_{\tau+1}}{y_\tau} = \frac{y_\tau}{y_{\tau-1}} = \frac{i_{\tau+1}}{i_\tau} = \frac{i_\tau}{i_{\tau-1}} = \frac{x_{\tau+1}}{x_\tau} = \frac{x_\tau}{x_{\tau-1}} = \mu^{\frac{1}{N}(\frac{1}{\alpha}-1)}.$$

B.2 Synchronized Implementation

B.2.1 N-sector, N-period Equilibrium

I will restrict attention to the N-cycle. If a firm receiving an idea in period $t-N+1$ is willing to postpone implementation to period N, then it will be willing to postpone to any period before that (*needs proof?*).

Period $t-N+1$. An innovation arrives to a firm in sector 1; it is saved and will be implemented in period N. In the previous period, $t-N$, all N sectors simultaneously implemented, which implies that now imitators can copy the new improved technology.

$$k_{t-N+2,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{\frac{i-1}{N}} n)^{\frac{1}{\alpha}}}{R_{t-N+1}^{\frac{1}{\alpha}}}, \ \text{where } i = 1, \dots \text{N}.$$

As $i_{t-N+1} = \sum_{i=1}^{N} \frac{k_{t-N+2,i}}{\mu^{\frac{i-1}{N}} n}$,

$$i_{T-N+1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\sum_{i=1}^{N} \mu^{\frac{i-1}{N}} n)^{\frac{1}{\alpha}-1}}{R_{T-N+1}^{\frac{1}{\alpha}}}.$$

As in $t-N$ all sectors implement their patents, the capital of the following period, $t-N+1$ remains unaffected compared to then, and only the investment level of period $T-N$ is affected. Then,

$$k_{t-N+1,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{(\frac{i-1}{N}-1)} n)^{\frac{1}{\alpha}}}{R_{t-N}^{\frac{1}{\alpha}}}, \text{ where } i = 1, \ldots \text{N}.$$

This combined with (79) implies production in $t-N+1$ is

$$y_{t-N+1} = \frac{(1-\alpha)^{\frac{1}{\alpha}-1} L(\sum_{i=1}^{N} \mu^{(\frac{i-1}{N}-1)} n)^{\frac{1}{\alpha}-1}}{R_{t-N}^{\frac{1}{\alpha}-1}}$$

Consumption follows from market-clearing:

$$x_{t-N+1} = (1-\alpha)^{\frac{1}{\alpha}} L(\sum_{i=1}^{N} \mu^{(\frac{i-1}{N})} n)^{\frac{1}{\alpha}-1} \left(\frac{\left(\frac{1}{\mu}\right)^{\frac{1}{\alpha}-1}}{(1-\alpha) R_{t-N}^{\frac{1}{\alpha}-1}} - \frac{1}{R_{t-N+1}^{\frac{1}{\alpha}}} \right). \tag{93}$$

It must be that $x_{t-N+1} > 0$.

Periods $(t-N+2)-(t-1)$**.** $t-N+j$ will be used throughout where $j = 2, \ldots, N-1$. Proceeding like before,

$$k_{t-N+j,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{(\frac{i-1}{N})} n)^{\frac{1}{\alpha}}}{R_{t-N+j-1}^{\frac{1}{\alpha}}}, \text{ where i=1, \ldots, N.}$$

This implies investment

$$i_{t-N+j-1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\sum_{i=1}^{N} \mu^{\frac{i-1}{N}} n)^{\frac{1}{\alpha}-1}}{R_{t-N+j-1}^{\frac{1}{\alpha}}}.$$

Production is

$$y_{t-N+j} = \frac{(1-\alpha)^{\frac{1}{\alpha}-1} L(\sum_{i=1}^{N} \mu^{(\frac{i-1}{N})} n)^{\frac{1}{\alpha}-1}}{R_{t-N+j-1}^{\frac{1}{\alpha}-1}},$$

while consumption follows from market-clearing:

$$x_{t-N+j} = (1-\alpha)^{\frac{1}{\alpha}} L(\sum_{i=1}^{N} \mu^{(\frac{i-1}{N})} n)^{\frac{1}{\alpha}-1} \left(\frac{1}{(1-\alpha) R_{t-N+j-1}^{\frac{1}{\alpha}-1}} - \frac{1}{R_{t-N+j}^{\frac{1}{\alpha}}} \right). \tag{94}$$

It must be that $x_{t-N+j} > 0$.

Period t**.** In t, patents are simultaneously implemented in all sectors. As shown previously, this does not affect the capital of the following period (only the ones' after it), but it affects the investment level of the implementation period.

$$k_{t+1,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{(\frac{i-1}{N})} n)^{\frac{1}{\alpha}}}{R_t^{\frac{1}{\alpha}}}, \text{ where i=1, \dots, N.}$$

Since, $i_t = \sum_{i=1}^{N} \frac{k_{t+1,i}}{\mu^{1+\frac{i-1}{N}} n}$, investment in t is

$$i_t = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\sum_{i=1}^{N} \mu^{(\frac{i-1}{N})} n)^{\frac{1}{\alpha}-1}}{\mu R_t^{\frac{1}{\alpha}}}.$$

Output is

$$y_t = \frac{(1-\alpha)^{\frac{1}{\alpha}-1} L(\sum_{i=1}^{N} \mu^{(\frac{i-1}{N})} n)^{\frac{1}{\alpha}-1}}{R_{t-1}^{\frac{1}{\alpha}-1}}.$$

Market-clearing in the consumption-good market implies

$$x_t = (1-\alpha)^{\frac{1}{\alpha}} L(\sum_{i=1}^{N} \mu^{(\frac{i-1}{N})} n)^{\frac{1}{\alpha}-1} \left(\frac{1}{(1-\alpha) R_{t-1}^{\frac{1}{\alpha}-1}} - \frac{1}{\mu R_t^{\frac{1}{\alpha}}} \right). \tag{95}$$

It must be that $x_t > 0$.

Period $t+1$**.** The following magnitudes will be generated:

$$k_{t+2,i} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\mu^{1+(\frac{i-1}{N})} n)^{\frac{1}{\alpha}}}{R_{t+1}^{\frac{1}{\alpha}}}, \text{ where i=1, \dots, N,}$$

$$i_{t+1} = \frac{(1-\alpha)^{\frac{1}{\alpha}} L(\sum_{i=1}^{N} \mu^{1+(\frac{i-1}{N})} n)^{\frac{1}{\alpha}-1}}{\mu R_{t+1}^{\frac{1}{\alpha}}},$$

$$y_{t+1} = \frac{(1-\alpha)^{\frac{1}{\alpha}-1} L(\sum_{i=1}^{N} \mu^{(\frac{i-1}{N})} n)^{\frac{1}{\alpha}-1}}{R_t^{\frac{1}{\alpha}-1}},$$

$$x_{t+1} = (1-\alpha)^{\frac{1}{\alpha}} L(\sum_{i=1}^{N} \mu^{(\frac{i-1}{N})} n)^{\frac{1}{\alpha}-1} \left(\frac{1}{(1-\alpha) R_t^{\frac{1}{\alpha}-1}} - \frac{\mu^{\frac{1}{\alpha}-1}}{R_{t+1}^{\frac{1}{\alpha}}} \right), \tag{96}$$

$x_{t+1} > 0$.

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